Adaptive Gauss Quadrature

Your work should be presented in the form of a typed report using clear and properly punctuated English. Where appropriate include full program listing and output. If you choose to work in a group of two, please turn in independently prepared reports.

1. Find an orthogonal polynomial p_4 of degree 4 such that

$$\int_{-1}^{1} q(x)p_4(x) = 0$$

for every polynomial $q(x)$ of degree 3 or less. You may use Maple and the Gram–Schmidt process as done in class. Alternatively, use your differential equation skills to find a polynomial solution to the differential equation

$$(1-x^2)y'' - 2xy' + 20y = 0.$$

2. The roots of $p_4(x)$ are real and lie in the interval $[-1,1]$. Use Newton's method with suitable starting points to find all four roots x_0, x_1, x_2 and x_3 as accurately as possible. Compute the residuals and the derivatives

$$p_4(x_j) \quad \text{and} \quad p_4'(x_j) \qquad \text{for} \qquad j = 1,2,3,4$$

and comment on the accuracy of your roots.

3. Find weights w_k for $k = 0,1,2,3$ such that

$$\int_{-1}^{1} x^j\,dx = \sum_{k=0}^{3} w_k x_k^j \qquad \text{for} \qquad j = 0,1,2,3.$$

Verify that

$$\int_{-1}^{1} x^j\,dx = \sum_{k=0}^{3} w_k x_k^j \qquad \text{for} \qquad j = 4,5,6,7.$$

4. Prove the equality

$$\int_a^b f(t)dt = \frac{b-a}{2}\int_{-1}^{1} f\Big(a + \frac{b-a}{2}(x+1)\Big)dx.$$

5. Define

$$G_4(a,b,f) = \frac{b-a}{2}\sum_{k=0}^{3} w_k f\Big(a + \frac{b-a}{2}(x_k+1)\Big)$$

We know from the verification in question 3 as well as the general theory of Gauss quadrature that

$$\Big|\int_a^b f(t)dt - G_4(a,b,f)\Big| = \mathcal{O}\big((b-a)^9\big) \qquad \text{as} \qquad b-a \to 0.$$

Let $c = (a + b)/2$ and use Richardson extrapolation to find α and β such that

$$R(a, b, f) = \alpha G_4(a, b, f) + \beta\big(G_4(a, c, f) + G_4(c, b, f)\big)$$

satisfies

$$\Big|\int_a^b f(t)dt - R(a, b, f)\Big| = \mathcal{O}\big((b - a)^{10}\big) \qquad \text{as} \qquad b - a \to 0.$$

6. Consider the adaptive quadrature rule given by

$$Q(a, b, f, \varepsilon) = \begin{cases} R(a, b, f) & \text{if } \big|G_4(a, b, f) - R(a, b, f)\big| < \varepsilon \\ \\ Q(a, c, f, \varepsilon/2) & \\ \qquad +Q(c, b, f, \varepsilon/2) & \text{otherwise} \end{cases}$$

and use this rule to approximate the improper integral

$$\int_0^1 f(t)dt \qquad \text{where} \qquad f(t) = \frac{1}{t}e^{-(\log t)^2}.$$

Since the exact value of the integral is $\sqrt{\pi}/2$ check that your approximation satisfies

$$\text{error} = \Big|Q(0, 1, f, \varepsilon) - \frac{\sqrt{\pi}}{2}\Big| \le \varepsilon \qquad \text{when} \qquad \varepsilon = 10^{-7}.$$

What happens to the above approximation when $\varepsilon = 10^{-p}$ for $p = 8, 9, 10, \ldots 15$?

7. [Extra Credit and for Math/CS 667] Write an optimized code implementing the adaptive quadrature method described in the previous question so that for each distinct interval $[a_i, b_i]$ the corresponding value of $G_4(a_i, b_i, f)$ is computed only once as the routine recurses.