

The Shu–Oscher Method

Your work should be presented in the form of a typed report using clear and properly punctuated English. Where appropriate include full program listing and output. If you choose to work in a group of two, please turn in independently prepared reports.

1. Consider the ordinary differential equation

$$y' = y^2 \cos(t), \quad y(0) = 0.8$$

on the interval $[0, 8]$. Find the exact solution.

2. Consider the three-stage Runge–Kutta method with Butcher tableau given by

$$\begin{array}{c|ccc} 0 & & & \\ 1 & 1 & & \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \\ \hline & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{array}$$

This is called the Shu–Oscher method. Use this method to approximate the solution of the differential equation given in part 1. Plot the exact solution and two different approximations, one for $h = 1/2$ and another for $h = 1/4$, on the same graph. Comment on the accuracy of the approximations.

3. Given $n \in \mathbf{N}$ define $h = 8/n$ and $t_j = hj$. Let y be the exact solution to the differential equation and y_j be the approximation of $y(t_j)$ obtained using the Shu–Oscher method. Define the error

$$E(h) = \max \{ |y_j - y(t_j)| : j = 0, 1, \dots, n \}.$$

If $E(h) \approx Kh^p$ for some K and some p we say the method is of order p . Numerically determine the order of the Shu–Oscher method by taking $n = 2^k$ for $k = 4, 5, \dots, 16$ for the differential equation in part 1. Plot the corresponding values of $E(h)$ versus h using log-log coordinates, note that $\log E(h) \approx p \log h + \log K$ and solve for p and K .

4. To obtain the most accurate approximate what size should h be taken? What happens if you take h even smaller? Can you find an h so small that $E(h)$ is 10 times larger than the minimal value? Why or why not?
5. [Extra Credit] Let `su3` represent one step of the Shu–Oscher method such that

$$y_{j+1} = \text{su3}(y_j, t_j, t_{j+1}).$$

Use Taylor’s theorem to show that the truncation error

$$\tau_h = y(t+h) - \text{su3}(y(t), t, t+h) = \mathcal{O}(h^4).$$

It is fine and recommended to use a computer algebra system such as Maple to assist in your calculations.

6. The Rössler System is a three dimensional ordinary differential equation of the form $du/dt = f(u)$ with a given initial condition $u(t_0) = u_0$ where $u(t) \in \mathbf{R}^3$ and

$$f(u) = \begin{bmatrix} -u_2 - u_3 \\ u_1 + au_2 \\ b + u_3(u_1 - c) \end{bmatrix} \quad \text{where} \quad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

with $a = b = 0.2$ and $c = 5.7$. Write a program to approximate $u(1)$ using the Shu–Osher method and the initial condition

$$u(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Approximate $u(1)$ by performing a convergence study taking h smaller and smaller to obtain an approximation good to at least 4 significant digits.

7. [Extra Credit] Is it possible to compute $u(10)$, $u(100)$ and $u(1000)$ to 4 significant digits. Explain, why or why not. Does using a different numerical scheme help?
8. Find the linear stability domain for the Shu–Osher method by considering the differential equation

$$y' = (a + ib)y$$

and determining what values of $z = h(a + ib)$ guarantee that the approximations $y_j \rightarrow 0$ as $j \rightarrow \infty$. Plot the domain and include the graph in your report.