Theorem. *Consider the ordinary differential equation initial-value problem*

$$\frac{dy}{dt} = f(y,t) \qquad \textit{with} \qquad y(t_0) = y_0$$

where $\|f_y(\xi,t)\| \le B$ *for* $\xi \in \mathbf{R}$ *and* $t \in [t_0, T]$. *Suppose there exists a unique solution* y *such that* $|y''(t)| \le A$ *for* $t \in [t_0, T]$. *Euler's method for approximating* y *is given by*

$$y_{k+1} = y_k + hf(y_k, t_k) \qquad \textit{where} \qquad t_k = t_0 + kh \quad \textit{and} \quad h = (T - t_0)/n.$$

Then $|y_n - y(T)| \to 0$ *as* $n \to \infty$.

Proof. Define $\varepsilon_k = y_k - y(t_k)$. Taylor's theorem implies there exists $c_k \in [t_k, t_{k+1}]$ such that

$$y(t_{k+1}) = y(t_k + h) = y(t_k) + hy'(t_k) + \frac{h^2}{2} y''(c_k)$$

and by the Mean Value Theorem there is ξ_k between y_k and $y(t_k)$ such that

$$f(y_k, t_k) - f(y(t_k), t_k) = f_y(\xi_k, t_k)\big(y_k - y(t_k)\big) = f_y(\xi_k, t_k)\varepsilon_k.$$

Note that $|\xi_k - y(t_k)| \le |\varepsilon_k|$ for $k = 1, \ldots, n$. Therefore

$$\begin{aligned}
\varepsilon_{k+1} &= y_{k+1} - y(t_{k+1}) = y_k + hf(y_k, t_k) - y(t_k) - hy'(t_k) - \frac{h^2}{2} y''(c_k) \\
&= \varepsilon_k + h\big(f(y_k, t_k) - f(y(t_k), t_k)\big) - \frac{h^2}{2} y''(c_k) \\
&= \big(1 + hf_y(\xi_k, t_k)\big)\varepsilon_k - \frac{h^2}{2} y''(c_k)
\end{aligned}$$

and consequently

$$|\varepsilon_{k+1}| \le (1 + hB)|\varepsilon_k| + \frac{h^2}{2} A.$$

Since $\varepsilon_0 = |y_0 - y(0)| = 0$, then by induction

$$\begin{aligned}
&|\varepsilon_1| \le \frac{h^2}{2} A, \qquad |\varepsilon_2| \le (1 + hB)\frac{h^2}{2} A + \frac{h^2}{2} A, \\
&|\varepsilon_3| \le \big((1 + hB)^2 + (1 + hB) + 1\big)\frac{h^2}{2} A, \\
&\quad \cdots \\
&|\varepsilon_n| \le \big((1 + hB)^{n-1} + \cdots + (1 + hB) + 1\big)\frac{h^2}{2} A, \\
&\quad = \Big(\frac{(1 + hB)^n - 1}{hB}\Big)\frac{h^2}{2} A = \big((1 + hB)^n - 1\big)\frac{hA}{2B}.
\end{aligned}$$

Now

$$\lim_{n\to\infty} (1 + hB)^n = \lim_{n\to\infty} \Big(1 + \frac{(T - t_0)B}{n}\Big)^n = e^{(T - t_0)B}$$

implies

$$|y_n - y(T)| = |\varepsilon_n| \to \big(e^{(T - t_0)B} - 1\big)\frac{0A}{2B} = 0 \qquad \text{as} \qquad n \to \infty.$$

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