Take Home Final for Math 467/667

Instructions: This is the take-home part of the final exam and must be worked independently. Among other things, that means no interactive messaging, email or posts to answer forums about the contents herein. You may, however, send email to your instructor if you find errors or have specific concerns about any of the problems. Please feel free to use your notes, the textbook, other books, your computer, programs such Maple and Mathematica, languages such as C and Julia as well as web searches to access existing online information and documentation. Please upload all work (including computer programs, scripts and output) related to your solutions for each problem to the appropriate link on the UNR WebCampus. You may upload one file for each question, so paste all of your work into one pdf before uploading it.

1. Consider the dot product and norm defined by

$$\mathtt{dp}(f,g)=\int_{-\infty}^{\infty} f(x)g(x)e^{-x^2}\,dx \qquad \text{and} \qquad \mathtt{nm}(f,f)=\sqrt{\mathtt{dp}(f,f)}.$$

   Find the polynomials $H_n(x)$ of degree n for $n=0,1,\ldots,5$ that are orthonormal with respect to the above dot product and norm.

2. It is known that the roots x_k of the polynomial $H_5(x)$ from above are given by

k	x_k
0	−2.020182870456086
1	−0.9585724646138185
2	0.0
3	0.9585724646138185
4	2.020182870456086

   Find the weights w_k for $k=0,\ldots,4$ such that the approximation

$$\int_{-\infty}^{\infty} f(x)e^{-x^2}\,dx \approx \sum_{k=0}^{4} w_k f(x_k)$$

   is exact when $f(x)$ is a polynomial of degree 9.

3. Given $h>0$ and $t_n=t_0+hn$, consider the three-stage Runge-Kutta method

$$\begin{aligned} k_1 &= f(t_n,y_n)\\ k_2 &= f(t_n+h/3,\,y_n+hk_1/3)\\ k_3 &= f(t_n+2h/3,\,y_n+2hk_2/3)\\ y_{n+1} &= y_n+h(k_1+3k_3)/4 \end{aligned}$$

for approximating as $y_n \approx y(t_n)$ the exact solution to $y' = f(t,y)$ with $y(t_0) = y_0$. Find the truncation error and the resulting order of the method.

4. Given $h > 0$ and $t_n = t_0 + hn$, consider the implicit trapezoid method

$$y_{n+1} = y_n + \frac{h}{2}\big(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\big)$$

for approximating as $y_n \approx y(t_n)$ the exact solution to $y' = f(t,y)$ with $y(t_0) = y_0$. Compute and draw a graph of the linear stability domain for this implicit method.

5. Please provide a signed statement indicating that you have read and understood the instructions for this take-home exam and that the work you have submitted represents your independent effort on all questions. For example, the single sentence "I have worked independently on this exam" accompanied by your signature is sufficient.