Working on Quadrature (approximating integrals)

$$\int_a^b f(\tau)\, w(\tau)\, d\tau \approx \sum_{j=1}^{\nu} b_j\, f(c_j)$$

weight function is part of the b_j's so it's been integrated

Given c_i's in $[a,b]$ we for b_j's such that

$$\int_a^b \tau^m\, w(\tau)\, d\tau = \sum_{j=1}^{\nu} b_j\, c_j^m \qquad \text{for } m=0,1,\ldots,\nu-1$$

As a result, for any polynomial p of degree $\nu-1$ we have

$$\int_a^b p(\tau)\, w(\tau)\, d\tau = \sum_{j=1}^{\nu} b_j\, p(c_j)$$

p is a polynomial of degree $\nu-1$

The polynomial interpolating theorem implies

$$\left| \int_a^b f(\tau)\, w(\tau)\, d\tau - \sum_{j=1}^{\nu} b_j\, f(c_j) \right| \le c \max\left\{ |f^{(\nu)}(\xi)| : \xi \in [a,b] \right\}$$

where

$$\boxed{c \le \int_a^b \left| \frac{q(\tau)}{\nu!}\, w(\tau) \right| d\tau} \qquad q(\tau) = (\tau - c_1)(\tau - c_2)\cdots(\tau - c_\nu)$$

As a result we said the quadrature formula was of order ν if it is exact for polynomials of degree $\nu-1$ or less.

How to choose the c_i's best? reasonably?

Idea of Chebycev: Choose the c_i's to minimize the error in the interpolating polynomials...

For example choose c_i's to minimize

$$\max\{|(\tau-c_1)(\tau-c_2)\cdots(\tau-c_\nu)| : \tau\in[a,b]\}$$

This provides the least uniform bound on the error in the interpolating polynomial.

or possibly...

$$\max\{|(\tau-c_1)(\tau-c_2)\cdots(\tau-c_\nu)|\, w(\tau) : \tau\in[a,b]\}$$

Alternatively since we're focused on quadrature, minimize

$$\int_a^b \left|\frac{q(\tau)}{\nu!}\, w(\tau)\right| d\tau = \int_a^b \frac{|(\tau-c_1)(\tau-c_2)\cdots(\tau-c_\nu)|}{\nu!}\, w(\tau)\, d\tau.$$

Idea of Gauss... Choose the c_i's to maximize the degree of the polynomial such that

$$\int_a^b p(\tau)\, w(\tau)\, d\tau = \sum_{j=1}^{\nu} b_j\, p(c_j)$$

Maximizing the order of the quadrature method... amazingly in this case focusing on only the order is a good idea...

Way to find maxima is through orthogonality... Define orthogonal polynomials using the inner product (dot product)

$$\langle f,g\rangle = \int_a^b f(\tau)g(\tau)\, w(\tau)\, d\tau$$

Norm based on this inner product

$$\|f\| = \sqrt{\langle f,f\rangle} = \left(\int_a^b f(\tau)^2\, w(\tau)\, d\tau\right)^{1/2}$$

Once there is an inner product one can make things orthogonal using Gram-Schmidt orthogonalization.

Given "vectors" $u_0, u_1, \ldots, u_{\nu-1}$ then define

$$z_0 = u_0 \qquad\qquad p_0 = z_0 / \|z_0\|$$

$$z_1 = u_1 - \langle p_0, u_1 \rangle p_0 \qquad\qquad p_1 = z_1 / \|z_1\|$$

$$z_2 = u_2 - \langle p_0, u_2 \rangle p_0 + \langle p_1, u_2 \rangle p_1 \qquad\qquad p_2 = z_2 / \|z_2\|$$

$$\vdots$$

$$z_{\nu-1} = u_{\nu-1} - \langle p_0, u_{\nu-1} \rangle p_0 - \cdots - \langle p_{\nu-2}, u_{\nu-1} \rangle p_{\nu-2}$$

$$p_{\nu-1} = z_{\nu-1} / \|z_{\nu-1}\|$$

Start by setting $u_0(\tau) = 1$, $u_1(\tau) = \tau$, $u_2(\tau) = \tau^2$ ... $u_{\nu-1}(\tau) = \tau^{\nu-1}$. Then run the Gram-Schmidt and obtain polynomial $p_0(\tau), \ldots, p_{\nu-1}(\tau)$ such that

$$\langle p_i, p_j \rangle = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

and also degree of p_i is i.

Now choose c_i's to be the roots of $p_\nu(\tau)$

orthogonal poly of degree ν.

Thus I have ν roots

$$c_1, c_2, \ldots, c_\nu$$

By lemma 3.2 in the book $c_i \in [a, b]$ for all i and they are distinct.

after choosing the c_i's as the roots of $P_\nu(\tau)$

choose the b_i's so that

$$\int_a^b \tau^m w(\tau)\,d\tau = \sum_{j=1}^{\nu} b_j c_j^m \qquad \text{for } m=0,1,\dots,\nu-1$$

The conclusion: The resulting Quadrature method is of order 2ν.

We'll prove this next time.