- Due to the snow days homework is accepted up to Sunday night.
- I'll post my solutions Monday morning.
- Exam is Thursday March 16

<u>Explicit RK methods:</u> Usual beginning point is the ODE

$$y' = f(t,y), \qquad y(t_0) = y_0$$

Setup a grid t_n of time steps to approximate a solution of the initial value problem...

$$\int_{t_n}^{t_{n+1}} y'(t)\,dt = \int_{t_n}^{t_{n+1}} f(t, y(t))\,dt$$

approximate this integral..

$$y(t_{n+1}) - y(t_n) \approx \underbrace{(t_{n+1} - t_n)}_{h} f\left(\frac{t_n + t_{n+1}}{2}, y\left(\frac{t_n + t_{n+1}}{2}\right)\right) \quad O(h^3)$$

This point is not on the grid... how to make a scheme?

Since $h \cdot O(h^2) = O(h^3)$ this substitution doesn't effect the order...

Idea: Interpolate between y_n and y_{n+1}.

$$y\left(\frac{t_n + t_{n+1}}{2}\right) \approx \frac{y_n + y_{n+1}}{2} \qquad \text{this is } O(h^2)$$

This is the midpoint method

$$y_{n+1} - y_n = (t_{n+1} - t_n) f\left(\frac{t_n + t_{n+1}}{2}, \frac{y_n + y_{n+1}}{2}\right)$$

Make grid uniform: Then $t_n = t_0 + hn$ and

$$y_{n+1} = y_n + h f\left(t_n + \frac{h}{2}, \frac{y_n + y_{n+1}}{2}\right)$$

This is an implicit scheme that's order 2 which means the truncation error is $O(h^3)$. Not an explicit RK method (yet).

~~Idea: Interpolate between y_n and y_{n+1}.~~

Idea: Use the ODE to approximate $y(t_n+\frac{h}{2})$

$$\int_{t_n}^{t_n+\frac{h}{2}} y'\,dt = \int_{t_n}^{t_n+\frac{h}{2}} f(t, y(t))\,dt$$

Use Euler's method...

$$y(t_n+\tfrac{h}{2}) \approx y(t_n) + \tfrac{h}{2} f(t_n, y(t_n)) \quad \text{to } O(h^2)$$

Substitute this into

again $h \cdot O(h^2) = O(h^3)$

$$y(t_{n+1}) - y(t_n) \approx h\, f(t_n+\tfrac{h}{2}, y(t_n+\tfrac{h}{2})) \quad \text{to } O(h^3)$$

Thus we get the 2^{nd} order method

$$y_{n+1} = y_n + h f(t_n+\tfrac{h}{2}, y_n + \tfrac{h}{2} f(t_n, y_n))$$

The idea of an RK method is substituting on method into another one... Each method is called an RK stage...

$$\xi_1 = y_n$$

$$\xi_2 = y_n + \tfrac{h}{2} f(t_n, y_n)$$ is an $O(h^2)$ approx of $y(t_n+\frac{h}{2})$

$$y_{n+1} = y_n + h\, f(t_n+\tfrac{h}{2}, \xi_2)$$

Abstract this with parameters ... but before that one more example...

Another example, the implicit trapezoid method

$$y_{n+1} = y_n + h\left(\frac{f(t_n, y_n) + f(t_{n+1}, y_{n+1})}{2}\right) \quad \text{is } O(h^3)$$

interpolation of f's rather than the y's.

Convert this to an explicit method of the same order but substituting a method of lesser order in for the implicit part.. Euler's method

$$\boxed{y_{n+1} = y_n + hf(t_n, y_n)} \quad \text{is } O(h^2)$$

Thus

$$y_{n+1} = y_n + h\left(\frac{f(t_n,y_n) + f(t_{n+1}, y_n + hf(t_n,y_n))}{2}\right)$$

This is another 2-stage RK method

$$\xi_1 = y_n$$

$$\xi_2 = y_n + hf(t_n, \xi_1) \quad \text{is an } O(h^2) \text{ approx of } y(t_n+h)$$

$$y_{n+1} = y_n + \frac{h}{2} f(t_n, \xi_1) + \frac{h}{2} f(t_n+h, \xi_2)$$

Abstract this with parameters

$$\xi_1 = y_n \qquad c_1 = 0$$

$$\xi_2 = y_n + h a_{21} f(t_n, \xi_1) \quad \text{is an } O(h^2) \text{ approx of } y(t_n + c_2 h)$$

note $c_2 = a_{21}$

$$y_{n+1} = y_n + h b_1 f(t_n + c_1 h, \xi_1) + h b_2 f(t_n + c_2 h, \xi_2) \quad \text{approx of } y(t_{n+1})$$

$b_1 + b_2 = 1$

And abstract the number of stages

approx of $y(t_n)$: $\xi_1 = y_n$ $\qquad c_1 = 0$

approx of $y(t_n+c_2h)$: $\xi_2 = y_n + h a_{21} f(t_n + c_1 h, \xi_1)$ $\qquad c_2 = a_{21}$

approx of $y(t_n+c_3h)$: $\xi_3 = y_n + h a_{31} f(t_n + c_1 h, \xi_1) + h a_{32} f(t_n + c_2 h, \xi_2)$ $\qquad c_3 = a_{31} + a_{32}$

$\vdots$

approx of $y(t_n+c_\nu h)$: $\xi_\nu = y_n + h \sum_{i=1}^{\nu-1} a_{\nu i} f(t_n + c_i h, \xi_i)$ $\qquad c_\nu = \sum_{i=1}^{\nu-1} a_{\nu i}$

$$y_{n+1} = y_n + h \sum_{j=1}^{\nu} b_j f(t_n + c_j h, \xi_j) \qquad \sum_{j=1}^{\nu} b_j = 1$$

Now solve for the parameters a's, b's and c's to optimize something: order, stability?, energy conservation, monotonicity, other physical constraints... memory usage, fewest number of stages...

Notation. RK Tableaux

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

a way to describe the parameters...

For example

$$\xi_1 = y_n \qquad c_1 = 0$$

$$\xi_2 = y_n + h f(t_n, \xi_1) \qquad a_{21} = 1 \qquad c_2 = a_{21} = 1$$

$$y_{n+1} = y_n + \frac{h}{2} f(t_n, \xi_1) + \frac{h}{2} f(t_n + h, \xi_2)$$

$$b_1 = \tfrac{1}{2} \qquad b_2 = \tfrac{1}{2}$$

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

Another example

$$\xi_1 = y_n \qquad c_1 = 0$$

$$\xi_2 = y_n + \frac{h}{2} f(t_n, y_n) \qquad a_{21} = \tfrac{1}{2} \qquad c_2 = \tfrac{1}{2}$$

$$y_{n+1} = y_n + h \cdot 0 \cdot f(t_n, \xi_1) + h f\left(t_n + \frac{h}{2}, \xi_2\right)$$

$$b_1 = 0 \qquad b_2 = 1$$

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ \hline & 0 & 1 \end{array}$$

note that once ξ_2 is computed we can forget about ξ_1... so this method saves memory...

Try to make a higher order method: Gauss Quadrature formula?

$$\int_0^1 f(\tau)\, d\tau \approx \frac{1}{2} f\left(\frac{1}{2} - \frac{1}{\sqrt{12}}\right) + \frac{1}{2} f\left(\frac{1}{2} + \frac{1}{\sqrt{12}}\right)$$

$$y' = f(t, y)$$

$$\int_{t_n}^{t_{n+1}} y'(t)\, dt = \int_{t_n}^{t_{n+1}} f(t, y(t))\, dt$$

$$y(t_{n+1}) - y(t_n) \approx \frac{h}{2} f\left(t_n + \left(\frac{1}{2} - \frac{1}{\sqrt{12}}\right)h, y\left(t_n + \left(\frac{1}{2} - \frac{1}{\sqrt{12}}\right)h\right)\right)$$

$$+ \frac{h}{2} f\left(t_n + \left(\frac{1}{2} + \frac{1}{\sqrt{12}}\right)h, y\left(t_n + \left(\frac{1}{2} + \frac{1}{12}\right)h\right)\right) \qquad O(h^5)$$

If I could figure out an $O(h^4)$ way to approximate

$$y\left(t_n+\left(\tfrac{1}{2}-\tfrac{1}{\sqrt{12}}\right)h\right) \text{ and } y\left(t_n+\left(\tfrac{1}{2}+\tfrac{1}{12}\right)h\right)$$

then I'd have an order 4 integrator in the end....

Maybe lower my ambition to begin with and look for an $O(h^3)$ approximation of

$$y\left(t_n+\left(\tfrac{1}{2}-\tfrac{1}{\sqrt{12}}\right)h\right) \text{ and } y\left(t_n+\left(\tfrac{1}{2}+\tfrac{1}{12}\right)h\right)$$

for a order 3 integrator in the end...

2 stages each

For example approximate these using any of the RK 2 2^{nd} order schemes to get $O(h^3)$ approximations and substitute the approximations into the Gauss formula to get an explicit RK integrator of order 3.

$2+2+1$ 5 stage RK method of order 3.

Thats a lot of stages... Solving for the parameters to maximize the order while minimizing the stages leads to a 3-stage RK method of order 3.