From HW3 ...

3.4 Restricting your attention to scalar autonomous equations $y' = f(y)$, prove that the ERK method with tableau

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

is of order 4.

$$\xi_1 = y_n \qquad c_1 = 0$$

$$\xi_2 = y_n + \frac{h}{2} f(t_n, \xi_1) \qquad c_2 = \frac{1}{2}$$

$$\xi_3 = y_n + 0 f(t_n, \xi_1) + \frac{h}{2} f(t_n + \frac{h}{2}, \xi_2) \qquad c_3 = \frac{1}{2}$$

$$\xi_4 = y_n + 0 f(t_n, \xi_1) + 0 f(t_n + \frac{h}{2}, \xi_2) + h f(t_n + \frac{h}{2}, \xi_3) \qquad c_4 = 1$$

$$y_{n+1} = y_n + h\left(\frac{1}{6} f(t_n, \xi_1) + \frac{1}{3} f(t_n + \frac{h}{2}, \xi_2) + \frac{1}{3} f(t_n + \frac{h}{2}, \xi_3) + \frac{1}{6} f(t_n + h, \xi_4)\right)$$

Find truncation error ... plug exact solution into the method and find the residual error...

$$\xi_1 = y(t_n)$$

$$\xi_2 = y(t_n) + \frac{h}{2} f(t_n, \xi_1)$$

$$\xi_3 = y(t_n) + 0 f(t_n, \xi_1) + \frac{h}{2} f(t_n + \frac{h}{2}, \xi_2)$$

$$\xi_4 = y(t_n) + 0 f(t_n, \xi_1) + 0 f(t_n + \frac{h}{2}, \xi_2) + h f(t_n + \frac{h}{2}, \xi_3)$$

$$y(t_{n+1}) = y(t_n) + h\left(\frac{1}{6} f(t_n, \xi_1) + \frac{1}{3} f(t_n + \frac{h}{2}, \xi_2) + \frac{1}{3} f(t_n + \frac{h}{2}, \xi_3) + \frac{1}{6} f(t_n + h, \xi_4)\right)$$

Truncation error

$$\tau_n = y(t_{n+1}) - y(t_n) - h\left(\frac{1}{6} f(t_n, \xi_1) + \frac{1}{3} f(t_n + \frac{h}{2}, \xi_2) + \frac{1}{3} f(t_n + \frac{h}{2}, \xi_3) + \frac{1}{6} f(t_n + h, \xi_4)\right)$$

$$\xi_1 = y(t_n)$$

$$\xi_2 = y(t_n) + \frac{h}{2} f(t_n, \xi_1) \approx y(t_n) + \overbrace{\frac{h}{2} f(t_n, y(t_n))}^{\text{same } y'(t_n)}$$

$$\xi_3 = y(t_n) + 0 f(t_n, \xi_1) + \frac{h}{2} f(t_n + \frac{h}{2}, \xi_2)$$

$$= y(t_n) + 0 f(t_n, \xi_1) + \frac{h}{2} \underbrace{f(t_n + \frac{h}{2}, y(t_n) + \frac{h}{2} f(t_n, y(t_n)))}_{\boxed{\text{is not } y'(t_n + \frac{h}{2})}}$$

So to use Taylor's theorem need to convert y' to f rather that f to y' since it's not always possible to convert f to y'.

What does this mean when computing?

$$\tau_n = \underbrace{y(t_{n+1})}_{} - y(t_n) - h\left(\frac{1}{6} f(t_n, \xi_1) + \frac{1}{3} f(t_n + \frac{h}{2}, \xi_2) + \frac{1}{3} f(t_n + \frac{h}{2}, \xi_3) + \frac{1}{6} f(t_n + h, \xi_4)\right)$$

need to Taylor series expand this

$$y(t_{n+1}) = y(t_n + h)$$

$$= y(t_n) + h y'(t_n) + \frac{h^2}{2} y''(t_n) + \frac{h^3}{3!} y'''(t_n) + \frac{h^4}{4!} y^{(4)}(t_n) + O(h^5)$$

convert these to f since can't do it the other way around...

$$= y(t_n) + h f(t_n, y(t_n)) + \frac{h^2}{2} y''(t_n) + \frac{h^3}{3!} y'''(t_n) + \frac{h^4}{4!} y^{(4)}(t_n) + O(h^5)$$

$y' = f(t, y(t))$ ← non autonomous — means f depends on time.

$$y'' = \frac{d}{dt} y' = \frac{d}{dt} f(t, y(t)) = f_t(t, y(t)) + f_y(t, y(t)) \frac{d}{dt} y(t)$$

$$= f_t(t, y(t)) + f_y(t, y(t)) f(t, y(t))$$

product

$$y''' = \frac{d}{dt} y'' = \frac{d}{dt}\Big(f_t(t,y(t)) + f_y(t,y(t))\, \underline{f(t,y(t))} \Big)$$

$$\begin{aligned} &= f_{tt}(t,y(t)) + f_{ty}(t,y(t))\, f(t,y(t)) \\ &\quad + \big[f_{yt}(t,y(t)) + f_{yy}(t,y(t))\, f(t,y(t)) \big]\, f(t,y(t)) \\ &\quad + f_y(t,y(t)) \big[f_t(t,y(t)) + f_y(t,y(t))\, f(t,y(t)) \big] \end{aligned}$$

scalar autonomous equations $\boxed{y' = f(y)}$,
tableau

Subclass of ODE's that correspond to repeatable physical experiments...

$$y' = f(y(t))$$

$$y'' = f_y(y(t))\, y'(t) = f_y(y(t))\, f(y(t))$$

$$y''' = f_{yy}(y(t))\, f(y(t))^2 + f_y(y(t))^2 f(y(t))$$

- Less terms to cancel but only considering the subclass of autonomous ODEs.

- Extra credit, try $y' = f(t,y(t))$ using a computer algebra system like Maple or Mathematica and check to order in general...

Chapter 4: Stability...

Compare to convergence

- Convergence takes $h \to 0$ while fixing the time interval $[t_0, T]$.
- Stability takes h fixed and then considers what happens as $t \to \infty$

Chapter 4.1 Example

Chapter 4.2 Definition of linear stability...

Stiffness ODEs which are stiff illustrate why stability is important to think about numerically.

$$y' = Ay \quad \text{where } y(t) \in \mathbb{R}^2 \text{ and } A = \begin{bmatrix} -100 & 1 \\ 0 & -\frac{1}{10} \end{bmatrix}$$

linear system of ODEs.

$$\begin{cases} y_1' = -100 y_1 + y_2 \\ y_2' = -\frac{1}{10} y_2 \end{cases}$$

written explicitly this system

Consider Eulers method to solve, $y' = f(t, y)$

$$f(t,y) = \begin{bmatrix} -100 y_1 + y_2 \\ -\frac{1}{10} y_2 \end{bmatrix} = Ay$$

Thus

$$y_{n+1} = y_n + h f(t_n, y_n) = y_n + hAy_n = (I + hA) y_n$$

by induction

$$y_n = (I + hA)^n y_0$$

What's the exact solution? $y' = Ay$ $\quad y(t) = e^{At} y_0$

Review math 330 (or Math 285) or not... solve eigenvalue–eigenvector problem $Ax = \lambda x$ to find e^{At} (or the solution to $y' = Ay$).

eigenvalues

upper triangular matrix so determinant is just the product on the diagonal.

$$\det(A - \lambda I) = \det \begin{bmatrix} -100-\lambda & 1 \\ 0 & -\frac{1}{10}-\lambda \end{bmatrix} = (-100-\lambda)(-\tfrac{1}{10}-\lambda)$$

$$= (\lambda+100)(\lambda+\tfrac{1}{10}) = 0$$

$$\lambda = -100 \quad \text{or} \quad \lambda = -\tfrac{1}{10}$$

eigenvectors

$\lambda = -100$

$$A - \lambda I = \begin{array}{cc} \text{free} & \text{pivot} \\ \left[\begin{matrix} 0 \\ 0 \end{matrix}\right. & \left.\begin{matrix} 1 \\ -\frac{1}{10}+100 \end{matrix}\right] \end{array}$$

nullspace of this $\quad x_1 = \text{free}$, $x_2 = \text{pivot}$

eigenvector... $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and any multiple of this...

$$\frac{-1000+1}{10} = \frac{-999}{10}$$

$\lambda = -\frac{1}{10}$

$$A - \lambda I = \begin{bmatrix} -100+\frac{1}{10} & 1 \\ 0 & 0 \end{bmatrix}$$

thus... $-\frac{999}{10} x_1 + x_2 = 0$

set $x_1 = 1$

$$x = \begin{bmatrix} 1 \\ \frac{999}{10} \end{bmatrix}$$

and any multiple...

Decomposition (factorization) of A:

$$A = VDV^{-1} \qquad V = \begin{bmatrix} 1 & 1 \\ 0 & \frac{999}{10} \end{bmatrix} \qquad D = \begin{bmatrix} -100 & 0 \\ 0 & -\frac{1}{10} \end{bmatrix}$$

conjugacy relation ...

$$e^{At} = e^{VDV^{-1}t} = V e^{Dt} V^{-1} = V\begin{bmatrix} e^{-100t} & 0 \\ 0 & e^{-\frac{1}{10}t} \end{bmatrix} V^{-1}$$

and

$$y(t) = e^{At} y_0 = V \begin{bmatrix} e^{-100t} & 0 \\ 0 & e^{-\frac{1}{10}t} \end{bmatrix} V^{-1} y_0$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c = V^{-1} y_0$$

$$y(t) = \begin{bmatrix} 1 & 1 \\ 0 & \frac{999}{10} \end{bmatrix} \begin{bmatrix} e^{-100t} & 0 \\ 0 & e^{-\frac{1}{10}t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-100t} + c_2 \begin{bmatrix} 1 \\ \frac{999}{10} \end{bmatrix} e^{-\frac{1}{10}t}$$

Use same trick to simplify $y_n = (I + hA)^n y_0$

$$(I + hA)^n = (I + hVDV^{-1})^n = (VV^{-1} + hVDV^{-1})^n$$

conjugacy relation

$$= \left[V\left(V^{-1} + hDV^{-1}\right)\right]^n = \left[V(I + hD)V^{-1}\right]^n$$

$$= V(I + hD)^n V^{-1} = V \begin{bmatrix} 1 - h100 & 0 \\ 0 & 1 - h/10 \end{bmatrix}^n V^{-1}$$

$$= V \begin{bmatrix} (1 - h100)^n & 0 \\ 0 & (1 - h/10)^n \end{bmatrix} V^{-1}$$

$$y_n = V \begin{bmatrix} (1 - h100)^n & 0 \\ 0 & (1 - h/10)^n \end{bmatrix} V^{-1} y_0$$

$$= V \begin{bmatrix} (1-h100)^n & 0 \\ 0 & (1-h/10)^n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$y_n = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} (1-h100)^n + c_2 \begin{bmatrix} 1 \\ \frac{999}{10} \end{bmatrix} (1-\frac{h}{10})^n$$

$$y(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-100t} + c_2 \begin{bmatrix} 1 \\ \frac{999}{10} \end{bmatrix} e^{-\frac{1}{10}t}$$

compare these next time...