1. Prove the following theorem on the convergence of Euler’s method:

**Theorem.** Consider the ordinary differential equation initial-value problem

\[
\frac{dy}{dt} = f(t, y) \quad \text{with} \quad y(t_0) = y_0
\]

where \(|f_y(t, \xi)| \leq B\) for \(\xi \in \mathbb{R}\) and \(t \in [t_0, T]\). Suppose there exists a unique solution \(y\) such that \(|y''(t)| \leq A\) for \(t \in [t_0, T]\). Then Euler’s method for approximating \(y\) given by

\[
y_{k+1} = y_k + hf(t_k, y_k) \quad \text{where} \quad t_k = t_0 + kh \quad \text{and} \quad h = \frac{T - t_0}{n}
\]

satisfies the limit \(|y_n - y(T)| \to 0\) as \(n \to \infty\).
2. Define

\[ L_h f = -\frac{15}{4h^3} \int_{-h}^{h} \left(1 - \frac{3t^2}{h^2}\right) f(t) dt \]

(i) Use Taylor’s theorem to show that \( L_h f = f''(0) + O(h^2) \).

(ii) Suppose \( f_\varepsilon \) is a function such that

\[ |f(t) - f_\varepsilon(t)| \leq \varepsilon \quad \text{for every} \quad t \in \mathbb{R}. \]

Show that \( |L_h f_\varepsilon - f''(0)| \leq 15\varepsilon/h^2 + O(h^2) \).
3. Consider the integral
\[
\int_{a}^{b} f(x) \, dx \quad \text{where} \quad a = 0, \quad b = 5 \quad \text{and} \quad f(x) = x \sin x.
\]
Further consider Simpson’s formula
\[
S(\alpha, \beta, f) = \frac{\beta - \alpha}{6} \left( f(\alpha) + 4f\left(\frac{\alpha + \beta}{2}\right) + f(\beta) \right)
\]
and the resulting quadrature method given by
\[
Q_N(a, b, f) = \sum_{j=0}^{N-1} S(x_j, x_{j+1}, f)
\]
where \( N \in \mathbb{N} \) and \( x_j = a + hj \) with \( h = (b - a)/N \).

(i) Use the rules of Calculus to find the exact value \( A \) of the integral.

(ii) Write a program to compute \( E_N = |Q_N(a, b, f) - A| \) for \( N = 2^p \) where \( p = 4, 5, \ldots, 16 \). Place your program and its output in the subdirectory `final` and use `submit final` to electronically submit your answer.

(iii) [Extra Credit] Plot \( E_N \) versus \( N \) using log-log scale to verify the order of Simpson’s quadrature method numerically. Submit your plot and the script used to make it in the same `final` subdirectory as used above.
4. Consider the solutions to the equation

\[ f(z) = 0 \quad \text{where} \quad f(z) = 2 \sin z + z^2 + 1 \]

and the application of Newton’s method

\[ z_{n+1} = g(z_n) \quad \text{where} \quad g(z) = z - f(z)/f'(z) \]

and \( z_0 \) is an initial guess to find the roots.

(i) Write a program to compute the resulting limits for initial guesses of the form \( z_0 = a + ib \) where \((a, b) \in [0, 4] \times [0, 4] \). Plot the basin of attraction for Newton’s method by coloring each point in \([0, 4] \times [0, 4] \) based on the limit. Place your program and its output in the subdirectory \texttt{final} and use \texttt{submit final} to electronically submit your answer.

(ii) Interpret the color of each point in your plot by explaining what it means. In particular, what limit value does each color correspond to?