In 1814 Gauss [2] developed a method for approximating definite integrals using an optimal quadrature formula of the form

\[
\int_a^b f(x) \, dx \approx \sum_{k=0}^{n-1} w_k f(x_k)
\]

that is exact for polynomials of degree \(2n - 1\). Since a \(2n - 1\) degree polynomial is determined by \(2n\) coefficients and the values of \(x_k\) and \(w_k\) for \(k = 0, 1, \ldots, n - 1\) represent \(2n\) parameters, the existence of such a formula seems reasonable. To overcome the difficulty of directly solving the resulting \(2n\) non-linear equations for \(2n\) unknowns, Gauss employed the family of orthogonal polynomials that were developed by Legendre [4] as solutions to differential equations. Extensions of these techniques were provided by Christoffel [1] in 1877 who obtained the existence and uniqueness of optimal quadratures for a general class of weighted integrals. Additional information may be found in Gautschi [3].

Orthogonal Polynomials

Rather than following Legendre who describes the orthogonal polynomials \(p_n\) of degree \(n\) on the interval \([-1, 1]\) as solutions to the differential equation

\[
(1 - x^2)y'' - 2xy' + n(n + 1)y = 0,
\]

we instead use the Gram–Schmidt orthogonalization process.

Consider the inner product and norm on the space of integrable functions defined by

\[
(f, g) = \int_{-1}^{1} f(x)g(x) \, dx \quad \text{and} \quad \|f\| = \sqrt{(f, f)}.
\]

The orthogonal polynomials

\[
\{ p_k : k = 0, 1, \ldots n \}
\]
may be obtained using the Gram–Schmidt orthogonalization procedure with respect to the above inner product and norm starting with the standard polynomial basis

\[ \{ x^k : k = 0, 1, \ldots n \}. \]

In particular, the orthogonal polynomials are given by

\[
\begin{align*}
    v_0 &= 1 \\
    v_1 &= x - (x, p_0)p_0 \\
    v_2 &= x^2 - (x^2, p_0)p_0 - (x^2, p_1)p_1 \\
    &\vdots \\
    v_n &= x^n - \sum_{k=0}^{n-1} (x^n, p_k)p_k
\end{align*}
\]

**Construction of Gauss Quadrature**

The points \( x_k \) and the weights \( w_k \) used in the approximation

\[
\int_{-1}^{1} f(x) \, dx \approx \sum_{k=0}^{n-1} w_k f(x_k)
\]

that we shall call Gauss quadrature are given as follows. Let \( x_k \) for \( k = 0, 1, \ldots, n - 1 \) be the \( n \) distinct roots to the orthogonal polynomial \( p_n \) of degree \( n \). Thus \( p_n(x_k) = 0 \) for \( k = 0, 1, \ldots, n - 1 \). We remark without proof that the \( x_k \)'s are real and moreover that \( x_k \in [-1, 1] \). Now, consider the system of \( n \) linear equations given by

\[
\int_{-1}^{1} x^j \, dx = \sum_{k=0}^{n-1} w_k x_k^j \quad \text{for} \quad j = 0, 1, \ldots, n - 1
\]

in the \( n \) unknowns \( w_k \) where \( k = 0, 1, \ldots n - 1 \). Since the \( x_j \)'s are distinct this system is non-singular. Therefore, there exists a unique solution for the \( w_k \)'s. This specifies the \( x_k \)'s and \( w_k \)'s in the Gauss quadrature formula.

**Accuracy of Gauss Quadrature**

In this section we prove Gauss quadrature is exact for polynomials of degree \( 2n - 1 \).

**Proof.** Let \( p \) be a polynomial of degree \( 2n - 1 \). Since the \( p_n \) has degree \( n \), the division algorithm implies there exist polynomials \( r \) and \( q \) of degree \( n - 1 \) such that

\[
p(x) = q(x)p_n(x) + r(x).
\]
Claim that
\[ \int_{-1}^{1} r(x) dx = \sum_{k=0}^{n-1} w_k r(x_k). \]

Write
\[ r(x) = \sum_{j=0}^{n-1} a_j x^j. \]

Then by the choice of \( w_k \)'s we have
\[ \int_{-1}^{1} r(x) dx = \sum_{j=0}^{n-1} a_j \int_{-1}^{1} x^j dx \]
\[ = \sum_{j=0}^{n-1} a_j \sum_{k=0}^{n-1} w_k x_k^j = \sum_{k=0}^{n-1} w_k \sum_{j=0}^{n-1} a_j x_k^j = \sum_{k=0}^{n-1} w_k r(x_k). \]

Since \( p_n \) is orthogonal to all polynomials of degree \( n - 1 \) or less and \( p_n(x_k) = 0 \), then
\[ \int_{-1}^{1} p(x) dx = \int_{-1}^{1} (q(x)p_n(x) + r(x)) dx = (q, p_n) + \int_{-1}^{1} r(x) dx \]
\[ = \int_{-1}^{1} r(x) dx = \sum_{k=0}^{n-1} w_k r(x_k) = \sum_{k=0}^{n-1} w_k (q(x_k) \cdot 0 + r(x_k)) \]
\[ = \sum_{k=0}^{n-1} w_k (q(x_k)p_n(x_k) + r(x_k)) = \sum_{k=0}^{n-1} w_k p(x_k). \]

This finishes the proof.

References
4. A. M. Legendre, Recherches sur la Figure des Planètes, Histoire de l’Académie Royale des Sciences, 1784, pp. 1–370.