Exercise 1.1 Apply the method of proof of Theorems 1.1 and 1.2 to prove the convergence of the implicit midpoint rule

$$y_{n+1} = y_n + hf\big(t_n + \tfrac{1}{2}h, \tfrac{1}{2}(y_n + y_{n+1})\big)$$

and of the theta method

$$y_{n+1} = y_n + h\big(\theta f(t_n, y_n) + (1-\theta) f(t_{n+1}, y_{n+1})\big).$$

Solution: These methods are for approximating the differential equation

$$y' = f(t, y) \qquad \text{such that} \qquad y(t_0) = y_0.$$

The goal is to show convergence as $h \to 0$ on an interval $[t_0, T]$ where $T > t_0$ is as large as desired. Divide the interval into a uniform grid by setting $t_n = t_0 + hn$ where $h = (T - t_0)/N$ is the step size and N is a positive integer indicating the number of subintervals.

In particular, we need to show that the maximum error on the interval

$$E_N = \max\{\, |e_n| : n = 0, \ldots, N \,\} \to 0 \qquad \text{as} \qquad h \to 0$$

Here $e_n = y_n - y(t_n)$ is the difference between the approximation given by the numerical scheme and the exact solution.

First consider the implicit midpoint rule and begin by computing the truncation error

$$\psi_n = y(t_{n+1}) - y(t_n) - hf\big(t_n + \tfrac{1}{2}h, \tfrac{1}{2}(y(t_n) + y(t_{n+1}))\big).$$

Upon setting $a = t_n + \frac{1}{2}h$ for convenience, Taylor's theorem then implies

$$y(t_{n+1}) = y(a) + \frac{h}{2}y'(a) + \frac{h^2}{8}y''(a) + \mathcal{O}(h^3)$$

and

$$y(t_n) = y(a) - \frac{h}{2}y'(a) + \frac{h^2}{8}y''(a) + \mathcal{O}(h^3).$$

Since $y'(a) = f(a, y(a))$, then

$$\psi_n = hf(a, y(a)) - hf\big(a, \tfrac{1}{2}(y(t_n) + y(t_{n+1}))\big) + \mathcal{O}(h^3).$$

Assuming f is Lipshitz continuous in the second variable with constant λ, further estimates yield

$$|\psi_n| \le h\lambda \big| y(a) - \tfrac{1}{2}(y(t_n) + y(t_{n+1}))\big| + \mathcal{O}(h^3).$$

Adding the Taylor series expansions for $y(t_n)$ and $y(t_{n+1})$ implies

$$y(t_n) + y(t_{n+1}) = 2y(a) + \mathcal{O}(h^2).$$

Consequently, $|\psi_n| \le h\lambda|\mathcal{O}(h^2)| + \mathcal{O}(h^3)$ and so $\psi_n = \mathcal{O}(h^3)$.

To prove convergence, define $e_n = y_n - y(t_n)$ and estimate e_{n+1} in terms of e_n. Thus,

$$\begin{aligned} e_{n+1} &= y_{n+1} - y(t_{n+1}) \\ &= y_n + hf\big(t_n + \tfrac{1}{2}h, \tfrac{1}{2}(y_n + y_{n+1})\big) - \psi_n - y(t_n) \\ &\qquad - hf\big(t_n + \tfrac{1}{2}h, \tfrac{1}{2}(y(t_n) + y(t_{n+1}))\big) \\ &= e_n + hf\big(t_n + \tfrac{1}{2}h, \tfrac{1}{2}(y_n + y_{n+1})\big) \\ &\qquad - hf\big(t_n + \tfrac{1}{2}h, \tfrac{1}{2}(y(t_n) + y(t_{n+1}))\big) + \mathcal{O}(h^3). \end{aligned}$$

Now, since

$$\begin{aligned} &|f\big(t_n + \tfrac{1}{2}h, \tfrac{1}{2}(y_n + y_{n+1})\big) - f\big(t_n + \tfrac{1}{2}h, \tfrac{1}{2}(y(t_n) + y(t_{n+1}))\big)| \\ &\qquad \le \lambda\big|\tfrac{1}{2}(y_n + y_{n+1}) - \tfrac{1}{2}(y(t_n) + y(t_{n+1}))\big| \le \tfrac{\lambda}{2}|e_n| + \tfrac{\lambda}{2}|e_{n+1}| \end{aligned}$$

it follows that

$$|e_{n+1}| \le |e_n| + \tfrac{h\lambda}{2}|e_n| + \tfrac{h\lambda}{2}|e_{n+1}| + \mathcal{O}(h^3).$$

Therefore, provided $h\lambda < 2$ we obtain

$$|e_{n+1}| \le \Big(\frac{1 + h\lambda/2}{1 - h\lambda/2}\Big)|e_n| + ch^3.$$

Assuming $e_0 = 0$, induction then results in

$$|e_n| \le \frac{\big(\frac{1+h\lambda/2}{1-h\lambda/2}\big)^n - 1}{\frac{1+h\lambda/2}{1-h\lambda/2} - 1} ch^3 \le \frac{1 - h\lambda/2}{\lambda}\Big\{\Big(\frac{1 + h\lambda/2}{1 - h\lambda/2}\Big)^n - 1\Big\}ch^2.$$

Estimating as

$$\frac{1+h\lambda/2}{1-h\lambda/2} = 1 + \frac{h\lambda}{1-h\lambda/2} \le \exp\Big(\frac{h\lambda}{1-h\lambda/2}\Big)$$

yields that

$$|e_n| \le \frac{1-h\lambda/2}{\lambda}\Big\{\exp\Big(\frac{hn\lambda}{1-h\lambda/2}\Big) - 1\Big\}ch^2.$$

Since $t_n = t_0 + hn$, then $hn = t_n - t_0 \le T - t_0$ when $n = 0, \ldots, N$. Therefore

$$|e_n| \le \frac{1-h\lambda/2}{\lambda}\Big\{\exp\Big(\frac{(T-t_0)\lambda}{1-h\lambda/2}\Big) - 1\Big\}ch^2.$$

The above bound does not depend on n. Consequently,

$$E_N \le \frac{1-h\lambda/2}{\lambda}\Big\{\exp\Big(\frac{(T-t_0)\lambda}{1-h\lambda/2}\Big) - 1\Big\}ch^2 \to 0 \quad \text{as} \quad h \to \infty.$$

Thus, the midpoint rule is convergent. Although we can infer the order of the method is $\mathcal{O}(h^2)$ from the truncation error $\psi_n = \mathcal{O}(h^3)$. The fact that $E_N = \mathcal{O}(h^2)$ can also be seen directly from the convergence proof.

If $h \to 0$ then at some point $h\lambda < 1$. Thus, $1 - h\lambda/2 > 1/2$ and one immediately obtains

$$E_N \le \frac{1}{\lambda}\Big\{\exp\big(2(T-t_0)\lambda\big) - 1\Big\}ch^2 = \mathcal{O}(h^2).$$

Which again shows the midpoint rule converges with $\mathcal{O}(h^2)$.

Next, consider the theta method. In this case the trucation error is

$$\psi_n = y(t_{n+1}) - y(t_n) - h\Big(\theta f(t_n, y(t_n)) + (1-\theta) f(t_{n+1}, y(t_{n+1}))\Big).$$

Taylor's theorem yields

$$y(t_{n+1}) = y(t_n) + hy'(t_n) + \frac{h^2}{2}y''(t_n) + \mathcal{O}(h^3)$$

as well as

$$y'(t_{n+1}) = y'(t_n) + hy''(t_n) + \mathcal{O}(h^2).$$

Use the differential equation to write

$$f(t_n, y(t_n)) = y'(t_n) \qquad \text{and} \qquad f(t_{n+1}, y(t_{n+1})) = y'(t_{n+1})$$

and substitute the result from Taylor's theorem to obtain

$$\begin{aligned}\psi_n &= y(t_n) + hy'(t_n) + \frac{h^2}{2}y''(t_n) + \mathcal{O}(h^3) - y(t_n) \\ &\quad - h\theta y'(t_n) - h(1-\theta)\big(y'(t_n) + hy''(t_n) + \mathcal{O}(h^2)\big) \\ &= h^2\Big\{\frac{1}{2} - \theta\Big\}y''(t_n) + \mathcal{O}(h^3).\end{aligned}$$

After noting the truncation error is $\mathcal{O}(h^3)$ when $\theta = 1/2$ but only $\mathcal{O}(h^2)$ in general, we write $\psi_n = \mathcal{O}(h^2)$ and procede with the convergence proof.

$$\begin{aligned}e_{n+1} &= y_{n+1} - y(t_{n+1}) \\ &= y_n + h\big(\theta f(t_n, y_n) + (1-\theta) f(t_{n+1}, y_{n+1})\big) - \psi_n - y(t_n) \\ &\qquad - h\big(\theta f(t_n, y(t_n)) + (1-\theta) f(t_{n+1}, y(t_{n+1}))\big) \\ &\le e_n + h\theta\big(f(t_n, y_n) - f(t_n, y(t_n))\big) \\ &\qquad + h(1-\theta)\big(f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y(t_{n+1}))\big) + \mathcal{O}(h^2).\end{aligned}$$

Since the Lipshitz property of f implies

$$\big|f(t_n, y_n) - f(t_n, y(t_n))\big| \le \lambda\big|y_n - y(t_n)\big| = \lambda|e_n|$$

and

$$\big|f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y(t_{n+1}))\big| \le \lambda\big|y_{n+1} - y(t_{n+1})\big| = \lambda|e_{n+1}|$$

it follows that

$$|e_{n+1}| \le |e_n| + h\lambda\theta|e_n| + h\lambda(1-\theta)|e_{n+1}| + \mathcal{O}(h^2).$$

Consequently,

$$|e_{n+1}| \le \Big(\frac{1 + h\lambda\theta}{1 - h\lambda(1-\theta)}\Big)|e_n| + ch^2.$$

Similar to before, assuming $e_0 = 0$ and using induction then results in

$$
\begin{aligned}
|e_n| &\leq \frac{\left(\frac{1+h\lambda\theta}{1-h\lambda(1-\theta)}\right)^n - 1}{\frac{1+h\lambda\theta}{1-h\lambda(1-\theta)} - 1} ch^2 \\
&= \frac{1-h\lambda(1-\theta)}{\lambda}\left\{\left(\frac{1+h\lambda\theta}{1-h\lambda(1-\theta)}\right)^n - 1\right\} ch.
\end{aligned}
$$

Estimating as

$$
\frac{1+h\lambda\theta}{1-h\lambda(1-\theta)} = 1 + \frac{h\lambda}{1-h\lambda(1-\theta)} \leq \exp\left(\frac{h\lambda}{1-h\lambda(1-\theta)}\right)
$$

and using the fact that $hn \leq T - t_0$ yields that

$$
\begin{aligned}
|e_n| &\leq \frac{1-h\lambda(1-\theta)}{\lambda}\left\{\exp\left(\frac{hn\lambda}{1-h\lambda(1-\theta)}\right) - 1\right\} ch \\
&\leq \frac{1-h\lambda(1-\theta)}{\lambda}\left\{\exp\left(\frac{(T-t_0)\lambda}{1-h\lambda(1-\theta)}\right) - 1\right\} ch.
\end{aligned}
$$

The above bound does not depend on n. Consequently,

$$
E_N \leq \frac{1-h\lambda(1-\theta)}{\lambda}\left\{\exp\left(\frac{(T-t_0)\lambda}{1-h\lambda(1-\theta)}\right) - 1\right\} ch \to 0
$$

as $h \to \infty$. Thus, the theta method is convergent. We can infer from the truncation error when $\theta = 1/2$ that the method is $\mathcal{O}(h^2)$ but when $\theta \neq 1/2$ that the theta method is only $\mathcal{O}(h)$. This latter fact may also be seen in the above convergence estimate.

Exercise 1.3 We solve the scalar linear system $y' = ay$ with $y(0) = 1$.

a Let $t_n = hn$ and show the continuous output method

$$u(t) = \frac{1 + \frac{1}{2}a(t - t_n)}{1 - \frac{1}{2}a(t - t_n)} y_n \quad \text{with} \quad t_n \le t \le t_{n+1}, \quad n = 0, 1, \ldots,$$

is consistent with the values of y_n and y_{n+1} which are obtained by the trapezoidal rule.

Solution: The trapezoid method is given by

$$y_{n+1} = y_n + \tfrac{1}{2}h[f(t_n, y_n) + f(t_{n+1}, y_{n+1})].$$

Substituting $f(t, y) = ay$ to obtain

$$y_{n+1} = y_n + \tfrac{1}{2}h[ay_n + ay_{n+1}]$$

and solving yields

$$y_{n+1} = \frac{1 + \frac{1}{2}ah}{1 - \frac{1}{2}ah} y_n.$$

Induction and the fact that $y_0 = 1$ then implies

$$y_n = \Big(\frac{1 + \frac{1}{2}ah}{1 - \frac{1}{2}ah}\Big)^n y_0 = \Big(\frac{1 + \frac{1}{2}ah}{1 - \frac{1}{2}ah}\Big)^n.$$

To show the continuous output method is consistent with the these values of y_n it is enough to show

$$\lim_{t \to t_n^+} u(t) = \lim_{t \to t_n^-} u(t).$$

Computing

$$\lim_{t \to t_n^+} u(t) = \lim_{t \to t_n^+} \frac{1 + \frac{1}{2}a(t - t_n)}{1 - \frac{1}{2}a(t - t_n)} y_n = y_n$$

along with

$$\begin{aligned}\lim_{t \to t_n^-} u(t) &= \lim_{t \to t_n^-} \frac{1 + \frac{1}{2}a(t - t_{n-1})}{1 - \frac{1}{2}a(t - t_{n-1})} y_{n-1} = \frac{1 + \frac{1}{2}a(t_n - t_{n-1})}{1 - \frac{1}{2}a(t_n - t_{n-1})} y_{n-1} \\ &= \Big(\frac{1 + \frac{1}{2}ah}{1 - \frac{1}{2}ah}\Big) y_{n-1} = y_n\end{aligned}$$

shows these two limits are equal.

b Demonstrate that u obeys the perturbed differential equation

$$u'(t) = au(t) + \frac{\frac{1}{4}a^3(t-t_n)^2}{[1-\frac{1}{2}a(t-t_n)]^2}y_n \quad \text{for} \quad t\in[t_n,t_{n+1}]$$

with initial condition $u(t_n) = y_n$. Thus, prove that

$$u(t_{n+1}) = e^{ha}\bigg[1+\frac{a^3}{4}\int_0^h \frac{e^{-\tau a}\tau^2 d\tau}{(1-a\tau/2)^2}\bigg]y_n.$$

Solution: Differentiating by the quotient rule yields

$$\begin{aligned}
u'(t) &= \frac{d}{dt}\Big(\frac{1+\frac{1}{2}a(t-t_n)}{1-\frac{1}{2}a(t-t_n)}\Big)y_n\\
&= \frac{\frac{1}{2}a(1-\frac{1}{2}a(t-t_n)) + \frac{1}{2}a(1+\frac{1}{2}a(t-t_n))}{(1-\frac{1}{2}a(t-t_n))^2}y_n\\
&= \frac{a}{(1-\frac{1}{2}a(t-t_n))^2}y_n\\
&= au(t) + a\bigg\{\frac{1}{(1-\frac{1}{2}a(t-t_n))^2}y_n - u(t)\bigg\}\\
&= au(t) + a\bigg\{\frac{1}{(1-\frac{1}{2}a(t-t_n))^2} - \frac{1+\frac{1}{2}a(t-t_n)}{1-\frac{1}{2}a(t-t_n)}\bigg\}y_n\\
&= au(t) + a\bigg\{\frac{1}{(1-\frac{1}{2}a(t-t_n))^2} - \frac{1-(\frac{1}{2}a(t-t_n))^2}{(1-\frac{1}{2}a(t-t_n))^2}\bigg\}y_n.
\end{aligned}$$

Therefore

$$u'(t) = au(t) + \frac{\frac{1}{4}a^3(t-t_n)^2}{(1-\frac{1}{2}a(t-t_n))^2}y_n$$

which establishes the differential equation.

Now multiply by the integrating factor e^{-at} to obtain

$$\frac{d}{dt}u(t)e^{-at} = e^{-at}\frac{\frac{1}{4}a^3(t-t_n)^2}{(1-\frac{1}{2}a(t-t_n))^2}y_n.$$

Integrating over the interval $[t_n, t]$ and using the fact that $u(t_n) = y_n$ yields

$$u(t)e^{-at} - y_n e^{-at_n} = \int_{t_n}^{t} e^{-as} \frac{\frac{1}{4}a^3(s-t_n)^2 ds}{(1-\frac{1}{2}a(s-t_n))^2} y_n$$

or equivalently

$$u(t) = y_n e^{a(t-t_n)} + e^{at} \int_{t_n}^{t} e^{-as} \frac{\frac{1}{4}a^3(s-t_n)^2 ds}{(1-\frac{1}{2}a(s-t_n))^2} y_n.$$

Substitute $\tau = s - t_n$ as

$$u(t) = y_n e^{a(t-t_n)} + e^{a(t-t_n)} \int_{0}^{t-t_n} e^{-a\tau} \frac{\frac{1}{4}a^3\tau^2 d\tau}{(1-\frac{1}{2}a\tau)^2} y_n$$

and finally take the limit $t \to t_{n+1}$ to obtain

$$y_{n+1} = u(t_{n+1}) = e^{ah}\left[1 + \frac{a^3}{4}\int_0^h \frac{e^{-a\tau}\tau^2 d\tau}{(1-\frac{1}{2}a\tau)^2}\right] y_n.$$

c Let $e_n = y_n - y(t_n)$ for $n = 0, 1, \ldots$. Show that

$$e_{n+1} = e^{ah}\left[1 + \frac{a^3}{4}\int_0^h \frac{e^{-a\tau}\tau^2 d\tau}{(1-\frac{1}{2}a\tau)^2}\right] e_n + \frac{a^3}{4} e^{at_{n+1}} \int_0^h \frac{e^{-a\tau}\tau^2 d\tau}{(1-\frac{1}{2}a\tau)^2}.$$

In particular, deduce that $a < 0$ implies that the error propagates subject to the inequality

$$|e_{n+1}| \le e^{ah}\left[1 + \frac{|a|^3}{4}\int_0^h e^{-a\tau}\tau^2 d\tau\right] |e_n| + \frac{|a|^3}{4} e^{at_{n+1}} \int_0^h e^{-a\tau}\tau^2 d\tau.$$

Solution: Begin by finding the exact solution to

$$y' = ay \qquad \text{such that} \qquad y(0) = 1.$$

Separation of variables and integrating yields

$$\int_{y_0}^{y(t)} \frac{dy}{y} = \int_{t_0}^{t} a\,dt \qquad \text{or} \qquad \log y_0 - \log y(t) = a(t - t_0).$$

Plugging in $t_0 = 0$ and $y_0 = 1$ then results in

$$\log y(t) = at \qquad \text{or} \qquad y(t) = e^{at}.$$

Note that

$$\begin{aligned} e_{n+1} = y_{n+1} - y(t_n) &= e^{ah}\left[1 + \frac{a^3}{4}\int_0^h \frac{e^{-a\tau}\tau^2 d\tau}{(1-\frac{1}{2}a\tau)^2}\right] y_n - e^{at_{n+1}} \\ &= e^{ah}\left[1 + \frac{a^3}{4}\int_0^h \frac{e^{-a\tau}\tau^2 d\tau}{(1-\frac{1}{2}a\tau)^2}\right] y_n - e^{ah} y(t_n) \\ &= e^{ah}\left[2 + \frac{a^3}{4}\int_0^h \frac{e^{-a\tau}\tau^2 d\tau}{(1-\frac{1}{2}a\tau)^2}\right] e_n + e^{at_{n+1}}\frac{a^3}{4}\int_0^h \frac{e^{-a\tau}\tau^2 d\tau}{(1-\frac{1}{2}a\tau)^2}. \end{aligned}$$

This is the equality that was to be shown.

Next, since $a < 0$ then $1 - \frac{1}{2}a\tau > 1$ and it follows that

$$\int_0^h \frac{e^{-a\tau}\tau^2 d\tau}{(1-\frac{1}{2}a\tau)^2} \le \int_0^h e^{-a\tau}\tau^2 d\tau.$$

Substituting this into the equation for e_{n+1} and estimating then yields

$$|e_{n+1}| \le e^{ah}\left[2 + \frac{|a|^3}{4}\int_0^h e^{-a\tau}\tau^2 d\tau\right]|e_n| + e^{at_{n+1}}\frac{|a|^3}{4}\int_0^h e^{-a\tau}\tau^2 d\tau$$

which is the desired bound on how the error propagates.

Exercise 1.4 Given $\theta \in [0,1]$, find the order of the method

$$y_{n+1} = y_n + hf\big(t_n + (1-\theta)h, \theta y_n + (1-\theta)y_{n+1}\big).$$

Solution: This appears to be a θ-parametarized version of the implicit midpoint rule considered earlier. As for that case set $a = t_n + (1-\theta)h$.

Begin by computing the truncation error

$$\psi_n = y(t_{n+1}) - y(t_n) - hf\big(t_n + (1-\theta)h, \theta y(t_n) + (1-\theta)y(t_{n+1})\big)$$

Noting $t_{n+1} = a + \theta h$ apply Taylor's theorem as

$$y(t_{n+1}) = y(a) + \theta h y'(a) + \frac{\theta^2 h^2}{2} y''(a) + \mathcal{O}(h^3)$$

and for $t_n = a - (1-\theta)h$ as

$$y(t_n) = y(a) - (1-\theta)hy'(a) + \frac{(1-\theta)^2 h^2}{2} y''(a) + \mathcal{O}(h^3).$$

Therefore

$$y(t_{n+1}) - y(t_n) = hy'(a) + \frac{(2\theta - 1)h^2}{2} y''(a) + \mathcal{O}(h^3).$$

The case $\theta = \frac{1}{2}$ was considered in Exercise 1.1 where it was shown the truncation error $\psi_n = \mathcal{O}(h^3)$ and the order of the method was $\mathcal{O}(h^2)$. What's left is the case when $\theta \neq \frac{1}{2}$. In this case

$$y(t_{n+1}) - y(t_n) = hy'(a) + \mathcal{O}(h^2).$$

Morever, since $y'(a) = f(a, y(a))$ we obtain

$$\psi_n = hf(a, y(a)) - hf\big(a, \theta y(t_n) + (1-\theta)y(t_{n+1})\big) + \mathcal{O}(h^2).$$

The Lipshitz condition on f implies

$$|\psi_n| \le h\lambda |y(a) - \theta y(t_n) + (1-\theta)y(t_{n+1})\big)| + \mathcal{O}(h^2)$$

and substituting the Taylor series for $y(t_n)$ and $y(t_{n+1})$ yields

$$\begin{aligned}|\psi_n| &\le h\lambda\Big| -\theta(1-\theta)hy'(a) + (1-\theta)\theta hy'(a) + \mathcal{O}(h^2)\big)\Big| + \mathcal{O}(h^2)\\ &\le h\lambda\Big|\mathcal{O}(h^2)\Big| + \mathcal{O}(h^2) = \mathcal{O}(h^2).\end{aligned}$$

Note it didn't matter that the $y'(a)$ terms cancelled. Since $\psi_n = \mathcal{O}(h^2)$ we conclude the method is order $\mathcal{O}(h)$ when $\theta \ne \frac{1}{2}$.

Though not required for this exercise, we procede to show the method is convergent. To this end note that

$$\begin{aligned}e_{n+1} &= y_{n+1} - y(t_{n+1})\\ &= y_n + hf\big(t_n + (1-\theta)h, \theta y_n + (1-\theta)y_{n+1}\big) - \psi_n\\ &\qquad - y(t_n) - hf\big(t_n + (1-\theta)h, \theta y(t_n) + (1-\theta)y(t_{n+1})\big).\end{aligned}$$

Consequently,

$$\begin{aligned}|e_{n+1}| &\le |e_n| + h\lambda\Big|\theta y_n + (1-\theta)y_{n+1} - \theta y(t_n) - (1-\theta)y(t_{n+1})\Big|\\ &\le (1 + h\lambda\theta)|e_n| + h\lambda(1-\theta)|e_{n+1}| + \mathcal{O}(h^2)\end{aligned}$$

and so

$$|e_{n+1}| \le \frac{1 + h\lambda\theta}{1 - h\lambda(1-\theta)}|e_n| + ch^2.$$

Now, assuming $e_0 = 0$, induction then results in

$$\begin{aligned}|e_n| &\le \frac{\big(\frac{1+h\lambda\theta}{1-h\lambda(1-\theta)}\big)^n - 1}{\frac{1+h\lambda\theta}{1-h\lambda(1-\theta)} - 1}ch^2\\ &\le \frac{1 - h\lambda(1-\theta)}{\lambda}\Big\{\big(\frac{1 + h\lambda\theta}{1 - h\lambda(1-\theta)}\big)^n - 1\Big\}ch.\end{aligned}$$

Take h small enough that $h\lambda(1-\theta) < 1/2$ and estimate as

$$\frac{1 + h\lambda\theta}{1 - h\lambda(1-\theta)} \le 1 + \frac{h\lambda}{1 - h\lambda(1-\theta)} \le 1 + 2h\lambda \le e^{2h\lambda}.$$

Then

$$|e_n| \le \frac{1}{\lambda}\Big\{e^{2hn\lambda} - 1\Big\}ch \le \frac{1}{\lambda}e^{2(T-t_0)\lambda}ch = \mathcal{O}(h).$$

Since this estimate is independent of n it follows that

$$E_N = \mathcal{O}(h) \qquad \text{as} \qquad h \to \infty.$$

This implies when $\theta \ne \frac{1}{2}$ the method is convergent with order $\mathcal{O}(h)$.

Exercise 1.5 Provided that f is analytic, it is possible to obtain from $y' = f(t, y)$ an expression for the second derivative of y, namely $y'' = g(t, y)$, where

$$g(t, y) = \frac{\partial f(t, y)}{dt} + \frac{\partial f(t, y)}{dy} f(t, y).$$

Find the orders of the methods

$$y_{n+1} = y_n + hf(t_n, y_n) + \tfrac{1}{2}h^2 g(t_n, y_n)$$

and

$$\begin{aligned} y_{n+1} = y_n &+ \tfrac{1}{2}h[f(t_n, y_n) + f(t_{n+1}, y_{n+1})] \\ &+ \tfrac{1}{12}h^2[g(t_n, y_n) - g(t_{n+1}, y_{n+1})]. \end{aligned}$$

Solution: The truncation error for the first method is

$$\begin{aligned} \psi_n &= y(t_{n+1}) - y(t_n) - hf(t_n, y_n) - \tfrac{1}{2}h^2 g(t_n, y_n) \\ &= y(t_n) + hy'(t_n) + \tfrac{1}{2}h^2 y''(t_n) + \mathcal{O}(h^3) \\ &\qquad - y(t_n) - hf(t_n, y_n) - \tfrac{1}{2}h^2 g(t_n, y_n) \\ &= \mathcal{O}(h^3). \end{aligned}$$

Therefore, the first method is $\mathcal{O}(h^2)$.

The truncation error for the second method is

$$\begin{aligned} \psi_n &= y(t_{n+1}) - y(t_n) - \tfrac{1}{2}h[f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))] \\ &\quad - \tfrac{1}{12}h^2[g(t_n, y(t_n)) - g(t_{n+1}, y(t_{n+1}))] \\ &= y(t_{n+1}) - y(t_n) - \tfrac{1}{2}h[y'(t_n) + y'(t_{n+1})] \\ &\quad - \tfrac{1}{12}h^2[y''(t_n) - y''(t_{n+1})] \end{aligned}$$

Taylor's theorem yields

$$\begin{aligned} y(t_{n+1}) &= y(t_n) + hy'(t_n) + \frac{h^2}{2}y''(t_n) + + \frac{h^3}{6}y^{(3)}(t_n) + \mathcal{O}(h^4) \\ y'(t_{n+1}) &= y'(t_n) + hy''(t_n) + \frac{h^2}{2}y^{(3)}(t_n) + \mathcal{O}(h^3) \\ y''(t_{n+1}) &= y''(t_n) + hy^{(3)}(t_n) + \mathcal{O}(h^2) \end{aligned}$$

consequently

$$
\begin{aligned}
\psi_n &= y(t_n) + hy'(t_n) + \frac{h^2}{2}y''(t_n) + \frac{h^3}{6}y^{(3)}(t_n) - y(t_n) \\
&\quad - \tfrac{1}{2}h\big[y'(t_n) + y'(t_n) + hy''(t_n) + \tfrac{h^2}{2}y^{(3)}(t_n)\big] \\
&\quad - \tfrac{1}{12}h^2\big[y''(t_n) - y''(t_n) - hy^{(3)}(t_n)\big] + \mathcal{O}(h^4) \\
&= \mathcal{O}(h^4).
\end{aligned}
$$

Therefore, the second method is $\mathcal{O}(h^3)$.

Showing these methods are convergent requires finding a Lipshitz condition for g. Such a condition follows from the fact that f was assumed analytic but is beyond the scope of the course and this exercise.