Kuramoto–Sivashinsky Equation

1. Consider the pattern formation equation

\[ u_t + uu_x + \nu u_{xx} + \mu u_{xxxx} = 0 \quad \text{with} \quad u(0, x) = u_0(x) \]

on the domain \([-1, 1]\) with periodic boundary conditions. Approximate \(u\) and \(u u_x\) using discrete Fourier series as

\[ u(x, t) \approx \sum_{n=-N/2+1}^{N/2} y_j(t) e^{inx} \quad \text{and} \quad (u u_x)(x, t) \approx \sum_{n=-N/2+1}^{N/2} B_n(y(t)) e^{inx} \]

where \(y = (y_0, \ldots, y_{N/2}, y_{-N/2+1}, \ldots, y_{-1})\) to obtain the system of ordinary differential equations

\[ \frac{dy_n}{dt} + B_n(y) - \nu \pi^2 n^2 y_n + \mu \pi^4 n^4 y_n = 0. \]

Note that \(B_n\) depends on \(t\) through \(y\) and may be computed using the subroutine developed in class for the viscous Burger equations. Write a program to integrate \(y_n\) on the interval \([0, T]\) using the split Euler scheme

\[ y_{n,j+1} = (y_{n,j} - h B_n(y_{n,j})) \exp(\nu \pi^2 n^2 h - \mu \pi^4 n^4 h) \]

where \(y_{n,j} = (y_{0,j}, \ldots, y_{N/2,j}, y_{-N/2+1,j}, \ldots, y_{-1,j})\), \(y_{n,j} \approx y_n(t_j)\) and \(t_j = jh\).

2. Set \(u_0(x) = \cos(\pi x) + \sin(3\pi x), \mu = 0.00001, \nu = 0.01, N = 128\) and \(h = T/J\) where \(T = 1\) and \(J = 16384\). Verify that \(u(0, T) \approx 0.32\). Draw a plot of \(u(x, T)\) versus \(x\).

3. For convenience define \(\alpha_n = \nu \pi^2 n^2 h - \mu \pi^4 n^4 h\) and modify your code to use the split RK2 scheme given by

\[ k_1,n = -hB_n(y_{n,j}) \]
\[ k_2,n = -he^{\alpha_n} B_n(p) \quad \text{where} \quad p_n = (y_{n,j} + k_1,n)e^{-\alpha_n} \]
\[ y_{n,j+1} = (y_{n,j} + (k_1,n + k_2,n)/2)e^{\alpha_n}. \]

Let \(U^h\) be the approximation of \(u(T)\) using the split RK2 method with step size \(h\). Graph \(\log \|U^h - U^{h/2}\|\) versus \(\log h\) where \(h = 2^{-j}\) for \(j = 6, \ldots, 16\) and

\[ \|U^h - U^{h/2}\| = \sqrt{\frac{2}{N} \sum_{\ell=-N/2+1}^{N/2} \left| U^h\left(\frac{2\ell}{N}\right) - U^{h/2}\left(\frac{2\ell}{N}\right) \right|^2} \]

to verify the order of convergence for the split RK2 method numerically. What happens if you take \(N = 256\)?

4. [Extra Credit] Repeat the previous question for the split RK4 method. Approximate the value of \(u(0, T)\) with as much precision as possible by increasing \(J\) and \(N\).