Adaptive Gauss Quadrature

Your work should be presented in the form of a typed report using clear and properly punctuated English. Where appropriate include full program listing and output. If you choose to work in a group of two, please turn in independently prepared reports.

1. Find an orthogonal polynomial $p_4$ of degree 4 such that

$$
\int_{-1}^{1} q(x)p_4(x) = 0
$$

for every polynomial $q(x)$ of degree 3 or less. You may use Maple and the Gram–Schmidt process as done in class. Alternatively, use your differential equation skills to find a polynomial solution to the differential equation

$$(1 - x^2)y'' - 2xy' + 20y = 0.$$ 

2. The roots of $p_4(x)$ are real and lie in the interval $[-1, 1]$. Use Newton’s method with suitable starting points to find all four roots $x_0, x_1, x_2$ and $x_3$ as accurately as possible. Compute the residuals and the derivatives

$$p_4(x_j) \text{ and } p_4'(x_j) \quad \text{for } j = 1, 2, 3, 4$$

and comment on the accuracy of your roots.

3. Find weights $w_k$ for $k = 0, 1, 2, 3$ such that

$$
\int_{-1}^{1} x^j dx = \sum_{k=0}^{3} w_k x_k^j \quad \text{for } j = 0, 1, 2, 3.
$$

Verify that

$$
\int_{-1}^{1} x^j dx = \sum_{k=0}^{3} w_k x_k^j \quad \text{for } j = 4, 5, 6, 7.
$$

4. Prove the equality

$$
\int_{a}^{b} f(t)dt = \frac{b-a}{2} \int_{-1}^{1} f\left(a + \frac{b-a}{2}(x+1)\right) dx.
$$

5. Define

$$G_4(a, b, f) = \frac{b-a}{2}\sum_{k=0}^{3} w_k f\left(a + \frac{b-a}{2}(x_k+1)\right)$$

We know from the verification in question 3 as well as the general theory of Gauss quadrature that

$$\left| \int_{a}^{b} f(t)dt - G_4(a, b, f) \right| = \mathcal{O}((b-a)^9) \quad \text{as } b - a \to 0.$$
Let \( c = (a + b)/2 \) and use Richardson extrapolation to find \( \alpha \) and \( \beta \) such that

\[
R(a, b, f) = \alpha G_4(a, b, f) + \beta (G_4(a, c, f) + G_4(c, b, f))
\]
satisfies

\[
\left| \int_a^b f(t)dt - R(a, b, f) \right| = O((b - a)^{10}) \quad \text{as} \quad b - a \to 0.
\]

6. Consider the adaptive quadrature rule given by

\[
Q(a, b, f, \varepsilon) = \begin{cases} 
R(a, b, f) & \text{if } |G_4(a, b, f) - R(a, b, f)| < \varepsilon \\
Q(a, c, f, \varepsilon/2) + Q(c, b, f, \varepsilon/2) & \text{otherwise}
\end{cases}
\]

and use this rule to approximate the improper integral

\[
\int_0^1 f(t)dt \quad \text{where} \quad f(t) = \frac{1}{t} e^{-(\log t)^2}.
\]

Since the exact value of the integral is \( \sqrt{\pi}/2 \) check that your approximation satisfies

\[
\text{error} = \left| Q(0, 1, f, \varepsilon) - \frac{\sqrt{\pi}}{2} \right| \leq \varepsilon \quad \text{when} \quad \varepsilon = 10^{-7}.
\]

What happens to the above approximation when \( \varepsilon = 10^{-p} \) for \( p = 8, 9, 10, \ldots 15 \)?

7. [Extra Credit and for Math/CS 667] Write an optimized code implementing the adaptive quadrature method described in the previous question so that for each distinct interval \([a_i, b_i]\) the corresponding value of \( G_4(a_i, b_i, f) \) is computed only once as the routine recurses.