Math 488/688 Exam 1 Version A

1. Solve the first order partial differential equation

$$\begin{cases} 2u_x + 3u_y = 5 - x \\ u(0, y) = \cos y. \end{cases}$$

Characteristics:

$\frac{dx}{dt} = 2 \qquad \frac{dy}{dt} = 3$

$x = 2t \qquad y = 3t + C$

Solve on characteristic:

$\frac{du}{dt} = 5 - 2t \qquad u = 5t - t^2 + D$

Thus

$u(2t, 3t + C) = 5t - t^2 + D$

Satisfy initial condition

$t = 0$

$u(0, C) = D = \sin C$

Thus

$u(2t, 3t + C) = 5t - t^2 + \sin C$

rewrite in terms of x and y.

$x = 2t \qquad t = \frac{x}{2}$

$y = 3t + C \qquad C = y - 3\frac{x}{2}$

Thus

$u(x, y) = \frac{5}{2}x - \frac{x^2}{4} + \sin\left(y - \frac{3x}{2}\right)$

2. Given the partial differential equation

$$\begin{cases} u_{tt} = 4u_{xx} - u \\ u(0,t) = 0 \\ u(6,t) = -\cos 3t \\ u(x,0) = 0 \\ u_t(x,0) = 0. \end{cases}$$

Write $u(x,t) = w(x,t) + v(x,t)$ and find a function w such that the resulting differential equation for v has homogeneous boundary conditions.

$w(x,t) = m(t)x + b(t)$

$w(0,t) = b(t) = 0 \qquad$ so $b(t) = 0$

$w(6,t) = 6m(t) = -\cos 3t$

so $m(t) = -\frac{1}{6}\cos 3t$

$w(x,t) = -\frac{x}{6}\cos 3t \qquad w(x,0) = -x/6$

$w_t(x,t) = \frac{x}{2}\sin 3t \qquad w_t(x,0) = 0$

$w_{tt}(x,t) = \frac{3x}{2}\cos 3t$

$w_{xx}(x,t) = 0$

$w_{tt} + v_{tt} = 4(w_{xx} + v_{xx}) - (v + w)$

$v(x,0) = u(x,0) - w(x,0)$

$v_t(x,0) = u_t(x,0) - w_t(x,0)$

$w(x,t) =$ $\boxed{-\frac{x}{6}\cos 3t}$

$v_{tt} = 4v_{xx} - v +$ $\boxed{-\frac{3x}{2}\cos 3t + \frac{x}{6}\cos 3t}$

$v(0,t) = 0 \qquad v(6,t) = 0$

$v(x,0) =$ $\boxed{\frac{x}{6}}$

$v_x(x,0) =$ $\boxed{\frac{1}{6}}$

$v_t(x,0) =$ $\boxed{0}$

3. Solve the second order partial differential equation

$$\begin{cases} u_t = u_{xx} + \cos \pi x \\ u_x(0,t) = 0 \\ u_x(3,t) = 0 \\ u(x,0) = 5. \end{cases}$$

Consider the homogeneous eq. $v_t = v_{xx}$ and use the separation of variables $v = XT$

$$XT' = X''T \quad \text{so} \quad \frac{T'}{T} = \frac{X''}{X} = k$$

Solve $X'' = kX$ subject to $X'(0) = 0$, $X'(3) = 0$

Case $k<0$: $X = A\cos\sqrt{|k|}\,x + B\sin\sqrt{|k|}\,x$

$X' = -A\sqrt{|k|}\sin\sqrt{|k|}\,x + B\sqrt{|k|}\cos\sqrt{|k|}\,x$

$X'(0) = B\sqrt{|k|} = 0$ implies $B = 0$

$X'(3) = -A\sqrt{|k|}\sin\sqrt{|k|}\,3 = 0$ implies $\sqrt{|k|}\,3 = n\pi$, $n = 1, 2, \ldots$

Case $k=0$: $X = Ax + B$

$X' = A$, $X'(0) = X'(3) = A = 0$ implies $A = 0$

so $X = B$ is a solution.

Case $k>0$: There are no non-trivial solutions.

Now, combine case $k<0$ and $k=0$ by taking $n = 0, 1, 2, \ldots$ in solutions of the form $X_n = A_n \cos\frac{n\pi x}{3}$.

Return to the non-homogeneous problem. Consider a solution of the form

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t) \cos\frac{n\pi x}{3}$$

write

$$\cos \pi x = \sum_{n=0}^{\infty} f_n \cos\frac{n\pi x}{3} \quad \text{gives } f_3 = 1 \text{ and } f_n = 0,\ n \neq 3.$$

Thus $f_n = \delta_{3,n}$

Similarly

$$u(x,0) = 5 = \sum_{n=0}^{\infty} a_n(0) \cos\frac{n\pi x}{3} \quad \text{gives } a_0(0) = 5 \text{ and } a_n(0) = 0,\ n \neq 0$$

Thus $a_n(0) = 5\,\delta_{0,n}$

Therefore substituting into the PDE we have

$$\sum_{n=0}^{\infty} a_n'(t) \cos\frac{n\pi x}{3} = \sum_{n=0}^{\infty} a_n(t)\left(-\frac{n^2\pi^2}{9}\right)\cos\frac{n\pi x}{3} + \sum_{n=0}^{\infty} \delta_{3,n}\cos\frac{n\pi x}{3}$$

Thus

$$a_n'(t) = -\frac{n^2\pi^2}{9}a_n(t) + \delta_{3,n}$$

It follows

$$\left(a_n e^{\frac{n^2\pi^2}{9}t}\right)' = \delta_{3,n} e^{\frac{n^2\pi^2}{9}t}$$

so

$$a_n(t)e^{\frac{n^2\pi^2}{9}t} - a_n(0) = \delta_{3,n}\int_0^t e^{\frac{n^2\pi^2}{9}s}ds = \frac{\delta_{3,n}9}{n^2\pi^2}\left(e^{\frac{n^2\pi^2 t}{9}} - 1\right)$$

or

$$a_n(t) = 5\delta_{0,n}e^{-\frac{n^2\pi^2 t}{9}} + \delta_{3,n}\cdot\frac{9}{n^2\pi^2}\left(1 - e^{-\frac{n^2\pi^2 t}{9}}\right)$$

Therefore

$$u(x,t) = \sum_{n=0}^{\infty}\left\{5\delta_{0,n}e^{-\frac{n^2\pi^2 t}{9}} + \delta_{3,n}\frac{9}{n^2\pi^2}\left(1-e^{-\frac{n^2\pi^2 t}{9}}\right)\right\}\cos\frac{n\pi x}{3}$$

$$= 5 + \frac{1}{\pi^2}\left(1 - e^{-\pi^2 t}\right)\cos\pi x$$

4. [Extra Credit and for Math 666] Express $x(x-1)$ on the interval $[0,1]$ as the series

$$x(x-1) = \sum_{n=1}^{\infty} F_n \sin n\pi x$$

by solving for the coefficents F_n.

$$F_n = 2\int_0^1 x(x-1)\sin n\pi x \, dx$$

$u = 2x(x-1)$ $\qquad du = (4x-2)dx$

$dv = \sin n\pi x \, dx$ $\qquad v = -\frac{1}{n\pi}\cos n\pi x$

$$= -\frac{2x(x-1)}{n\pi}\cos n\pi x \Big|_0^1 + \int_0^1 \frac{4x-2}{n\pi}\cos n\pi x \, dx$$

$u = 4x-2$ $\qquad du = 4dx$

$dv = \frac{1}{n\pi}\cos n\pi x \, dx$ $\qquad v = \frac{1}{n^2\pi^2}\sin n\pi x$

$$= \frac{4x-2}{n^2\pi^2}\sin n\pi x \Big|_0^1 - \int_0^1 \frac{4}{n^2\pi^2}\sin n\pi x \, dx$$

$$= \frac{4}{n^3\pi^3}\cos n\pi x \Big|_0^1 = \frac{4}{n^3\pi^3}\left((-1)^n - 1\right)$$