Math 488/688 Exam 2 Version B

1. Solve the first order partial differential equation

$$\begin{cases} xu_x + 2u_y = -yu \\ u(x,0) = x. \end{cases}$$

$$\frac{dx}{dt} = x, \quad \frac{dy}{dt} = 2, \quad \frac{du}{dt} = -yu$$

$$\frac{dxe^{-t}}{dt} = 0, \quad xe^{-t} = x_0, \quad x = Ce^t, \quad y = 2t$$

$$\frac{du}{dt} = -2tCe^t$$

$$u = -2C\int te^t dt = -2C\int t\,de^t = -2C\left(te^t - \int e^t dt\right)$$

$$= -2C(te^t - e^t) = 2C(1-t)e^t + D$$

Let $t=0$, then

$$u(C,0) = C = 2C + D, \qquad D = -C$$

$$u(Ce^t, 2t) = 2C(1-t)e^t - C$$

$$t = y/2, \quad x = Ce^{y/2}, \quad C = xe^{-y/2}$$

$$u(x,y) = 2xe^{-y/2}(1 - y/2)e^{y/2} - xe^{-y/2}$$

$$= 2x(1 - y/2) - xe^{-y/2}$$

2. Solve the second order partial differential equation

$$\begin{cases} u_t = u_{xx} \\ u(0,t) = 1 \\ u(1,t) = 1 \\ u(x,0) = 1 + \sin 2\pi x + 3\sin 5\pi x. \end{cases}$$

$u = w + v$

$w = a(t)x + b(t)$

$w(0,t) = b(t) = 1$, $b(t) = 1$

$w(1,t) = a(t) + 1 = 1$, $a(t) = 0$

Now v satisfies $v_t = v_{xx}$

$v(0,t) = v(1,t) = 0$

$v(x,0) = 1 - 1 + \sin 2\pi x + 3\sin 5\pi x$

seperation of variables:

$XT' = X''T$ $\qquad \frac{T'}{T} = \frac{X''}{X} = k$

Therefore $X'' = kX$, $X(0) = X(1) = 0$

Case $k<0$, $X = A\cos\sqrt{|k|}x + B\sin\sqrt{|k|}x$

$X(0) = A = 0$, $A = 0$

$X(1) = B\sin\sqrt{|k|} = 0$, $\sqrt{k} = n\pi$, $n = 1, 2, \ldots$

case $k \geq 0$ there are no nontrivial solutions.

Now $T' = -n^2\pi^2 T$ so $T = e^{-n^2\pi^2 t}$ and series solution is of the form

$$v(x,t) = \sum_{n=1}^{\infty} B_n \sin n\pi x \, e^{-n^2\pi^2 t}$$

$v(x,0) = \sin 2\pi x + 3\sin 5\pi x$ implies

$B_2 = 1$, $B_5 = 3$ and $B_n = 0$ for $n \neq 1$ and $n \neq 5$.

Therefore

$$v(x,t) = \sin 2\pi x \, e^{-4\pi^2 t} + 3\sin 5\pi x \, e^{-25\pi^2 t}$$

3. Solve the second order partial differential equation

$$\begin{cases} u_{xx} + 4u_{yy} = 0 \\ u(0, y) = 0 \\ u(1, y) = 0 \\ u(x, 0) = 0 \\ u(x, 1) = \sin(\pi x). \end{cases}$$

Separation of variables $X''Y + 4Y''X = 0$

$$\frac{X''}{X} + \frac{4Y''}{Y} = 0 \quad , \quad \frac{X''}{X} = -\frac{4Y''}{Y} = k .$$

Therefore $X'' = kX$, $X(0) = X(1) = 0$.

case $k<0$ then $X = A\sin\sqrt{|k|}\,x + B\cos\sqrt{|k|}\,x$

$X(0) = B$, $B = 0$

$X(1) = A\sin\sqrt{|k|} = 0$ $\sqrt{|k|} = n\pi$, $n = 1, 2, \ldots$

case $k \geq 0$ there are no non-trivial solutions.

Now $4Y'' = -kY = n^2\pi^2 Y$, $Y(0) = 0$

$Y = C\sinh\frac{n\pi}{2}y + D\cosh\frac{n\pi}{2}y$

$Y(0) = D$. $D = 0$

Therefore the series solution is of the form

$$u(x,y) = \sum A_n \sin n\pi x \sinh\frac{n\pi y}{2}$$

$$u(x,1) = \sum A_n \sin n\pi x \sinh\frac{n\pi}{2} = \sin\pi x$$

Equating coefficients yield

$A_n = 0$ for $n \neq 1$

$A_1 \sinh\frac{\pi}{2} = 1$ so $A_1 = \dfrac{1}{\sinh\frac{\pi}{2}}$

Thus

$$u(x,y) = \left(\frac{1}{\sinh\frac{\pi}{2}}\right)(\sin\pi x)\left(\sinh\frac{\pi y}{2}\right)$$

4. [Math 688 and Extra Credit] Let

$$f(x) = \begin{cases} x & \text{for } x \le 1 \\ 2-x & \text{for } x > 1. \end{cases}$$

Express $f(x)$ on the interval $[0,2]$ as the series

$$f(x) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi x}{2}\right)$$

by solving for the coefficents F_n.

$$F_n = \int_0^2 f(x) \sin\frac{n\pi x}{2}\, dx$$

$$= \int_0^1 x \sin\frac{n\pi x}{2}\, dx + \int_1^2 (2-x) \sin\frac{n\pi x}{2}\, dx = I_1 + I_2$$

where

$$I_1 = \frac{-2}{n\pi} \int_0^1 x\, d\cos\frac{n\pi x}{2} = -\frac{2}{n\pi}\left(x \cos\frac{n\pi x}{2}\Big|_0^1 - \int_0^1 \cos\frac{n\pi x}{2}\, dx \right)$$

$$= \frac{-2}{n\pi}\left(\cos\frac{n\pi}{2} - \frac{2}{n\pi} \sin\frac{n\pi x}{2}\Big|_0^1 \right) = \frac{-2}{n\pi}\left(\cos\frac{n\pi}{2} - \frac{2}{n\pi}\sin\frac{n\pi}{2} \right)$$

and

$$I_2 = \frac{-2}{n\pi} \int_1^2 (2-x)\, d\cos\frac{n\pi x}{2} = -\frac{2}{n\pi}\left((2-x)\cos\frac{n\pi x}{2}\Big|_1^2 + \int_1^2 \cos\frac{n\pi x}{2}\, dx \right)$$

$$= \frac{-2}{n\pi}\left(-\cos\frac{n\pi}{2} + \frac{2}{n\pi} \sin\frac{n\pi x}{2}\Big|_1^2 \right) = \frac{-2}{n\pi}\left(-\cos\frac{n\pi}{2} - \frac{2}{n\pi}\sin\frac{n\pi}{2} \right)$$

And so

$$F_n = I_1 + I_2 = \frac{8}{n^2\pi^2} \sin\frac{n\pi}{2}$$