LESSON 36

A Nonhomogeneous Dirichlet Problem (Green's Function)

PURPOSE OF LESSON: To show how a nonhomogeneous Dirichlet problem can be solved by the Green's function approach (the impulse-response function). This important technique *resolves* the right-hand side of the equation (generally thought of as an input of some kind) into a *continuum* of impulses (delta functions or point inputs) at the different points of the domain. The *response* to each of these impulses is then found (Green's function or the impulse-response function), and then they are *summed* (integrated) to give the overall response.

A common problem in applied mathematics is to find the potential in some region of space in response to a forcing term $f(x,y)$ acting *inside* the region. In *electrostatics*, the potential (volts) in a region D is sought in response to a charge density $f(x,y)$ throughout that region. A typical example would be to find the potential inside a circle in two dimensions that satisfies (Poisson's equation with zero BC)

$$
(36.1) \qquad \begin{array}{ll} \text{PDE} & u_{rr} + \dfrac{1}{r}u_r + \dfrac{1}{r^2}u_{\theta\theta} = f(r,\theta) \\ \text{BC} & u(1,\theta) = 0 \qquad 0 \le \theta \le 2\pi \end{array}
$$

Note that we have chosen the boundary values to be zero. If we wanted to solve the general case, where both the equation and BC were nonhomogeneous, we could add the Poisson integral formula from Lesson 33 to the solution from this lesson.

In order to gain a little intuition about nonhomogeneous differential equations, let's consider graphing the solution to the following Poisson's equation:

$$
\begin{array}{llll} \text{PDE} & \nabla^2 u = -q & 0 < r < 1 & (q \text{ a positive constant}) \\ \text{BC} & u(1,\theta) = 0 & 0 \le \theta \le 2\pi & \end{array}
$$

Here, the potential (temperature if you like) is fixed at zero on the boundary, and the Laplacian of u is always equal to $-q$ inside the circle. Since $\nabla^2 u(p)$ measures the difference between $u(p)$ and the average of its neighbors, Poisson's equation says that the surface $u(r,\theta)$ will always be concave down, so to speak. In other words, it will look like a thin membrane fixed at the boundary that was continuously being pushed up by a stream of air from below. If the right-hand side were a function $f(x,y)$ that changed over the domain, then the concavity at each point would change.

We now get to the major part of this lesson: to introduce Green's function and solve equation (36.1).

First, however, we must introduce the notion of potential due to *point sources and sinks*.

Potentials from Point Sources and Sinks

In solving a nonhomogeneous linear equation, it is sufficient to solve the equation with a point source, since we can find the solution to the general problem by summing the responses to point sources. Our goal here is to find the potential in some region of space due to a point source (or sink). We can interpret these points in a variety of ways. In heat flow, we could think of a source as a point where heat is created and a sink as a point where it is destroyed. On the other hand, in electrostatics, a point source would be a single positive charge (proton), while a sink would be a single negative charge (electron). In any case, whatever the interpretation, we will now find the potential $u(r)$ in two dimensions that depends on a single point source (the potential in three dimensions is left as a problem).

Suppose we have a single point source of magnitude $+q$ located at the origin. It is clear that the heat (or whatever) will flow outward along radial lines, and, hence, if we compute the total outward flux across a circle of radius r, we have the situation described in figure 36.1

$$\text{Total outward flux across the circle} = -\int_0^{2\pi} u_r(r)r\,d\theta$$

$$= -2\pi r u_r(r)$$

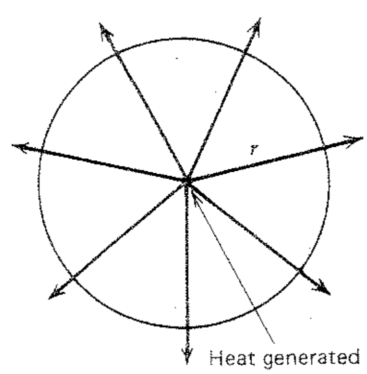


FIGURE 36.1 Radial flow of heat due to a point source.

But the outward flux must be equal to the heat generated within the circle (conservation of energy), and so we have

$$-2\pi r u_r(r) = q$$

Solving this simple differential equation for $u(r)$, we get

$$u(r) = \frac{-q}{2\pi} \ln r = \frac{q}{2\pi} \ln \frac{1}{r}$$

See Figure 36.2.

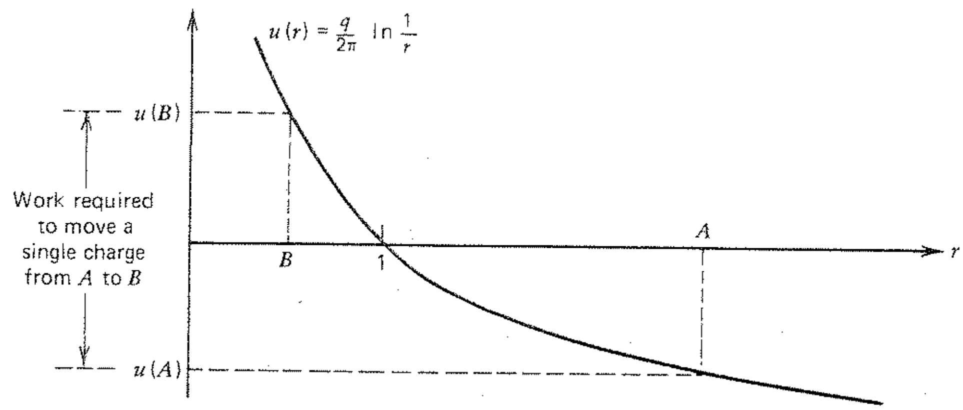


FIGURE 36.2 Potential due to a point source in two dimensions.

In terms of electrostatics, the potential difference $u(B) - u(A)$ represents the work needed to move a single positive charge from A to B (Figure 36.2). A *sink*, on the other hand, is represented by a *negative* source, and so a sink with magnitude $-q$ would give rise to a potential field

$$u(r) = \frac{-q}{2\pi} \ln \frac{1}{r}$$

This completes the discussion of potential due to point charges; we are now in position to solve the nonhomogeneous equation by means of Green's function.

Poisson's Equation inside a Circle

We will now solve the important problem

$$(36.2) \qquad \begin{aligned} &\text{PDE} \quad u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = f(r,\theta) \qquad 0 < r < 1 \\ &\text{BC} \quad u(1,\theta) = 0 \qquad 0 \leq \theta \leq 2\pi \end{aligned}$$

The Green function technique (impulse-response method) consists of two steps:

1. Finding the potential $G(r,\theta,\rho,\phi)$ at (r,θ), which we force to be zero on the boundary and which is due to a single charge (magnitude 1) at (ρ,ϕ)

2. Summing the individual responses $G(r,\theta,\rho,\phi)$ weighted by the right-hand side (charge density) $f(r,\theta)$ over all (ρ,ϕ) in the circle to get the solution

$$u(r,\theta) = \int_0^{2\pi} \int_0^1 G(r,\theta,\rho,\phi)\, f(\rho,\phi)\, \rho\, d\rho\, d\phi$$

We now find the impulse response $G(r,\theta,\rho,\phi)$ for our problem.

Finding the Potential Response $G(r,\theta,\rho,\phi)$

We first replace the right-hand side $f(r,\theta)$ by a point source of magnitude $+1$ at an arbitrary point (ρ,ϕ). Mathematically, we call a point source an impulse function (or delta function) and represent it by $\delta(r - \rho, \theta - \phi)$. We interpret this delta function as a function of r and θ that is zero for all points except at (ρ,ϕ), where the unit charge is located. In terms of forces, we could interpret the delta function as a point force of magnitude $+1$ at (ρ,ϕ). The idea now is to find the potential response (which we force to be zero on the boundary) due to a single point charge. This function is called the **impulse response function** (or **Green's function**), and it is the *response* at (r,θ) to a single *source* at (ρ,ϕ). The difficulty in finding this function is due to the fact that it must vanish on the boundary. If we didn't require zero, then the problem would be easy, since we already know that

$$\frac{1}{2\pi} \ln \frac{1}{r}$$

is the potential due to a charge at (ρ,ϕ) [where r is the distance from the charge (ρ,ϕ)].

Physically, finding $G(r,\theta,\rho,\phi)$ corresponds to one of the following:

1. Finding the equilibrium temperature inside the circle with a heat source at (ρ,ϕ) and the boundary temperature fixed at zero.
2. Finding the height of a stretched membrane fixed at zero on the boundary but pulled up to a great height at (ρ,ϕ).
3. Finding the electrostatic potential inside the circle due to a single positive charge at (ρ,ϕ) with the boundary potential grounded to zero.

We will now find Green's function; it will look something like Figure 36.3.

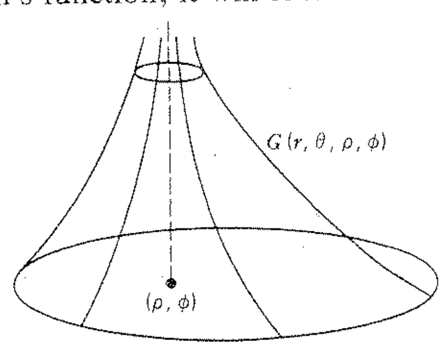


FIGURE 36.3 Green's function $G(r,\theta,\rho,\phi)$ due to a source at (ρ,ϕ).

Steps for Finding The Solution

STEP 1 Since the function

$$\frac{1}{2\pi} \ln \frac{1}{R}$$

is the potential at $P = (r,\theta)$ due to a single unit charge at $Q = (\rho,\phi)$ (where R is the *distance* from P to Q), the only thing left to do is modify the function so that it is zero on the boundary.

STEP 2 Physicists know from experiments that the potential field due to positive and negative charges placed a given distance apart give rise to *circles of constant potential* (Figure 36.4).

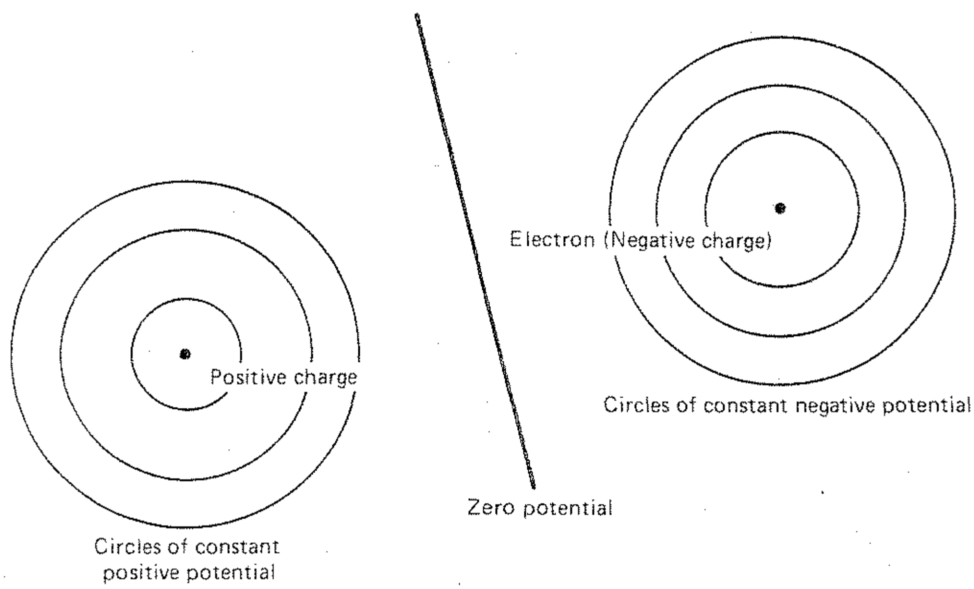


FIGURE 36.4 Potential field due to two oppositely charged particles.

So the strategy in finding Green's function is to place another charge (negative) *outside* the circle at such a point that the potential due to both is *constant* on the circle $r = 1$. We can then *subtract* this constant value to obtain a zero potential on the boundary. It is obvious now that this potential will satisfy our desired properties for $G(r,\theta,\rho,\phi)$. The big question is, of course, where do we place the *negative charge* outside the circle, so that the potential on the boundary is constant? Without going into the details, we can show rather easily that if the negative charge is placed at $\overline{Q} = (\overline{\rho},\overline{\phi}) = (1/\rho,\phi)$, then the *potential*

$$u(r,\theta) = \frac{1}{2\pi} \ln 1/R - \frac{1}{2\pi} \ln 1/\overline{R}$$

due to the two charges will be constant on the circle $r = 1$. The variables R and

$\overline{R}$ are the distances from the two charges to (r,θ). In fact, the constant potential on the circle $r = 1$ can easily be shown to be

$$\frac{-1}{2\pi} \ln \rho \qquad \text{(A positive constant)}$$

See Figure 36.5

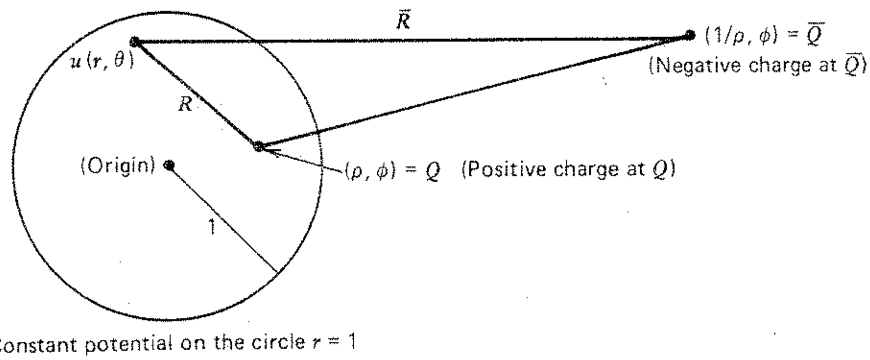


FIGURE 36.5 Charges at Q and $\bar{Q}$ giving rise to constant potential at $r = 1$.

With these steps in mind, we construct Green's function

$$G(r,\theta,\rho,\phi) = \frac{1}{2\pi} \ln 1/R - \frac{1}{2\pi} \ln 1/\overline{R} + \frac{1}{2\pi} \ln \rho \tag{36.3}$$

- $\frac{1}{2\pi} \ln 1/R$: Potential due to positive charge at Q
- $\frac{1}{2\pi} \ln 1/\overline{R}$: Potential due to negative charge at $\overline{Q}$
- $\frac{1}{2\pi} \ln \rho$: Subtracting the constant potential on the boundary

where

$$R = \sqrt{r^2 - 2r\rho \cos(\theta - \phi) + \rho^2}$$

$$\overline{R} = \sqrt{r^2 - 2\frac{r}{\rho} \cos(\theta - \phi) + 1/\rho^2}$$

(These two formulas are just trigonometric formulas for the distance between two points in polar coordinates.) To find the solution to our original problem, we merely superimpose the impulse functions; this brings us to the final step.

STEP 3 Superposition of the impulse responses. This step is easy; we just write

$$u(r,\theta) = \int_0^{2\pi} \int_0^1 G(r,\theta,\rho,\phi) f(\rho,\phi) \rho \, d\rho \, d\phi$$

or

$$(36.4) \qquad \boxed{u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \ln(\rho\overline{R}/R) f(\rho,\phi)\rho \, d\rho \, d\phi}$$

This is Green's function solution of Poisson's equation inside a circle. If we were given the charge density f(r,θ), we could evaluate this integral numerically.

NOTES

1. It is also possible to solve

$$\begin{array}{lll} \text{PDE} & \nabla^2 u = 0 & 0 < r < 1 \\ \text{BC} & u(1,\theta) = g(\theta) & 0 \le \theta \le 2\pi \end{array}$$

by means of the *Green's function* approach. In this case, the solution is

$$u(r,\theta) = \int_0^{2\pi} \frac{\partial G}{\partial r}(r,\theta,1,\phi) g(\phi) \, d\phi$$

which, if we compute $\partial G/\partial r$ (a rather tedious computation), gives

$$(36.5) \qquad u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2}\right] g(\phi) \, d\phi$$

which is the Poisson integral formula we found in Lesson 33.

2. The solution to the general Dirichlet problem

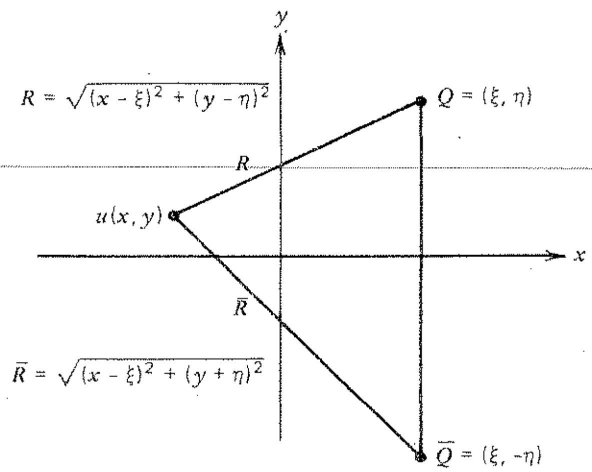
$$\begin{array}{lll} \text{PDE} & \nabla^2 u = f(r,\theta) & 0 < r < 1 \\ \text{BC} & u(1,\theta) = g(\theta) & 0 < \theta \le 2\pi \end{array}$$

would be the sum of equations (36.4) and (36.5).

3. We can solve many problems in different domains by means of the Green function approach. However, we must find a new Green function for each domain and each new equation and finding Green's function is not always easy.
4. To actually evaluate solution (36.4) for most sources $f(r,\theta)$, we must resort to numerical integration on a computer.

PROBLEMS

1. Find the potential due to a point source in three dimensions.
2. Find Green's function $G(x,y,\xi,\eta)$ for Laplace's equation in the upper-half plane $y > 0$. In other words, find the potential in the upper-half plane at the point (x,y) (zero on the boundary $y = 0$) due to a point charge at (ε,η). See the following figure.



HINT If we place a negative charge at $\overline{Q} = (\xi, -\eta)$, then it's clear that the potential field on the line $y = 0$ due to the two charges at Q and $\overline{Q}$ is zero. Hence, Green's function would be the resultant field due to these two charges.

3. Using the results of problem 2, what is the solution to Poisson's equation $\nabla^2 u = -k$ in the upper-half plane with zero BC?
4. How would you go about constructing Green's function for the first quadrant $x > 0$, $y > 0$?
5. An alternative approach to solving Poisson's equation that works sometimes is the following; suppose you want to solve:

$$\begin{array}{lll} \text{PDE} & \nabla^2 u = 1 & 0 < r < 1 \\ \text{BC} & u(1,\theta) = \sin\theta & 0 \leq \theta \leq 2\pi \end{array}$$

Start by trying to find any particular solution of $\nabla^2 u = 1$ by substituting

$$u_p(r,\theta) = Ar^2$$

into the differential equation and solving for the constant.

After finding a particular solution $u_p(r,\theta)$, consider letting $u = w + u_p$ and ask the question, what boundary-value problem will $w(r,\theta)$ satisfy? After you determine this, solve for $w(r,\theta)$. Finally, what is the answer $u(r,\theta)$ of the original problem? Does it check? Look at the answer carefully; what is the interpretation of each term?