2.5 $c^2 u_{xx} = u_{tt} + \mu u_t \quad x \in (0, L)$

$u(0,t) = 0 \qquad u(L,t) = 0$

$u(x,0) = \sin(\pi x / L)$

$u_t(x,0) = 0$

Separation of variables

$$c^2 X'' T = X T'' + \mu X T' = X(T'' + \mu T')$$

$$\frac{X''}{X} = \frac{T'' + \mu T'}{c^2 T} = k$$

$X'' = kX \qquad X(0) = 0 \qquad X(L) = 0$

$k < 0 \qquad X = A \cos\sqrt{|k|}x + B \sin\sqrt{|k|}x$

$X(0) = A = 0$

$X(L) = B \sin\sqrt{|k|}L = 0$

$\sqrt{|k|}L = n\pi \qquad n = 1, 2, \ldots$

$\sqrt{|k|} = n\pi / L$

$k = -n^2\pi^2 / L^2$

$$T'' + \mu T' + \frac{n^2\pi^2 c^2}{L^2} T = 0$$

Substitute $T = e^{rt}$ to obtain characteristic equ

$$r^2 + \mu r + \left(\frac{n\pi c}{L}\right)^2 = 0$$

discriminant is $\mu^2 - 4\left(\frac{n\pi c}{L}\right)^2 < 0$ under the assumption that μ^2 is very small. Thus the solution is

$$T = e^{-\frac{\mu}{2}t}\left(C \cos\sqrt{\left(\frac{2n\pi c}{L}\right)^2 - \mu^2}\ \tfrac{t}{2} + D \sin\sqrt{\left(\frac{2n\pi c}{L}\right)^2 - \mu^2}\ t\right)$$

General solution is of the form

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\, e^{-\frac{\mu}{2}t}\left(C_n \cos\sqrt{\ }\ \# \ t + D_n \sin\sqrt{\ }\ \# \ t\right)$$

To satisfy the initial conditions:

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} = \sin\frac{\pi x}{L}$$

implies $C_1 = 1$ and $C_2 = C_3 = C_4 = \cdots = 0$. To satisfy the condition

$$u_t(x,t) = \sum_{n=1}^{\infty} \left(\sin\frac{n\pi x}{L}\right)\left(-\frac{\mu}{2}\right) e^{-\frac{\mu}{2}t} \left(\quad \right)$$

$$+ \sum_{n=1}^{\infty} \left(\sin\frac{n\pi x}{L}\right) e^{-\frac{\mu}{2}t} \left(-C_n \tfrac{1}{2}\sqrt{\;}\ \sin\tfrac{1}{2}\sqrt{\;}\, t + D_n \tfrac{1}{2}\sqrt{\;}\ \cos\tfrac{1}{2}\sqrt{\;}\, t\right)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \left(\sin\frac{n\pi x}{L}\right)\left[\left(-\frac{\mu}{2}\right) C_n + D_n \tfrac{1}{2}\sqrt{\left(\frac{2n\pi c}{L}\right)^2 - \mu^2}\,\right] = 0$$

Therefore $D_n = \dfrac{\mu}{\sqrt{\;}} C_n$ and in particular

$$D_1 = \frac{\mu}{\sqrt{\left(\frac{2\pi c}{L}\right)^2 - \mu^2}}, \quad D_2 = D_3 = D_4 = \cdots = 0$$

Thus only $n=1$ term survives and

$$u(x,t) = \left(\sin\frac{\pi x}{L}\right)\left(e^{-\frac{\mu}{2}t}\right)\left(\cos\tfrac{1}{2}\sqrt{\left(\frac{2\pi c}{L}\right)^2 - \mu^2}\, t + \frac{\mu}{\sqrt{\left(\frac{2\pi c}{L}\right)^2 - \mu^2}} \sin\tfrac{1}{2}\sqrt{\left(\frac{2\pi c}{L}\right)^2 - \mu^2}\, t\right)$$