#5

$$u_{xx} + u_{yy} = \sin \pi x$$
$$u(x,0)=0$$
$$u(x,1)=0$$
$$u(0,y)=0$$
$$u(1,y)=\sin \pi y$$

Solve first the homogeneous equation

$$v_{xx} + v_{yy} = 0$$
$$v(x,0)=0$$
$$v(x,1)=0$$
$$v(0,y)=0$$

with separation of variables.

$$X''Y + XY'' = 0$$
$$X''Y = -XY''$$
$$-\frac{X''}{X} = \frac{Y''}{Y} = k$$

Thus $\quad Y'' = kY \;,\; Y(0)=0 \;,\; Y(1)=0$

$k<0 \qquad Y = A\cos\sqrt{|k|}\,y + B\sin\sqrt{|k|}\,y$

$$Y(0) = A = 0 \qquad A=0$$
$$Y(1) = B\sin\sqrt{|k|} = 0$$
$$\sqrt{|k|} = n\pi$$
$$k = -n^2\pi^2 \;,\; n=1,2,\ldots$$

we look for solution of the form

$$u(x,y) = \sum_{n=1}^{\infty} b_n(x)\sin n\pi y$$

write

$$\sin \pi x = \sum_{n=1}^{\infty} f_n(x)\sin n\pi y$$

$$f_n(x) = 2\int_0^1 \sin\pi x \sin n\pi y\, dy = \frac{2}{n\pi}\sin\pi x \cos n\pi y \Big|_0^1$$
$$= \frac{-2}{n\pi}(\sin\pi x)\left((-1)^n - 1\right)$$

Thus

$$\sum_{n=1}^{\infty} b_n''(x)\sin n\pi y - n^2\pi^2 \sum_{n=1}^{\infty} b_n(x)\sin n\pi y = \sum_{n=1}^{\infty} \frac{2}{n\pi}(\sin \pi x)((-1)^n-1)\sin n\pi y$$

so

$$b_n''(x) - n^2\pi^2 b_n(x) = \frac{2}{n\pi}(\sin \pi x)((-1)^n-1), \quad b_n(0)=0$$

To find a particular solution try

$$P_n(x) = C_n \sin \pi x$$
$$P_n'(x) = \pi C_n \cos \pi x$$
$$P_n''(x) = -\pi^2 C_n \sin \pi x$$

$$-\pi^2 C_n \sin \pi x - n^2\pi^2 C_n \sin \pi x = \frac{-2}{n\pi}(\sin \pi x)((-1)^n-1)$$

$$C_n = \frac{2((-1)^n-1)}{\pi^3 n(1+n^2)}$$

To find the general solution

$$H_n''(x) = n^2\pi^2 H_n(x), \quad H_n(0) = b_n(0) - P_n(0) = 0$$

$$H_n(x) = F_n e^{n\pi x} + G_n e^{-n\pi x}$$

$$H_n(0) = F_n + G_n = 0 \qquad G_n = -F_n$$

$$H_n(x) = 2F_n \sinh n\pi x$$

Thus $\quad b_n = \dfrac{2((-1)^n-1)}{\pi^3 n(1+n^2)}\sin \pi x + 2F_n \sinh n\pi x$

and

$$u(x,y) = \sum_{n=1}^{\infty}\left(\frac{2((-1)^n-1)}{\pi^3 n(1+n^2)}\sin \pi x + 2F_n \sinh n\pi x\right)\sin n\pi y$$

$$u(1,y) = \sum_{n=1}^{\infty} 2F_n \sinh n\pi \sin n\pi y = \sin \pi y$$

so $\quad F_1 = 1/(2\sinh \pi), \quad F_n = 0 \text{ for } n \neq 1$ and

$$u(x,y) = \frac{\sinh \pi x}{\sinh \pi}\sin \pi y + \sum_{n=1}^{\infty} \frac{2((-1)^n-1)}{\pi^3 n(1+n^2)}\sin \pi x \sin n\pi y$$