#1 $u_t = u_{xx}$

$u(0,t) = 0$

$u(1,t) = \frac{1}{e}\cos 2\pi t$

$u(x,0) = xe^{-x}$

Write $u = w + v$ where w satisfies $w(0,t) = 0$, $w(1,t) = \frac{1}{e}\cos 2\pi t$.

Thus $w(x,t) = a(t)x + b(t)$

$w(0,t) = b(t) = 0$

$w(1,t) = a(t) = \frac{1}{e}\cos 2\pi t$

so that

$$w(x,t) = \frac{x}{e}\cos 2\pi t$$

$$w_t = -\frac{2\pi x}{e}\sin 2\pi t$$

$$w_{xx} = 0$$

Therefore v satisfies

$$v_t = v_{xx} + \frac{2\pi x}{e}\sin 2\pi t$$

$$v(0,t) = v(1,t) = 0$$

$$v(x,0) = xe^{-x} - \frac{x}{e} = x(e^{-x} - e^{-1})$$

Seperation of variables gives

$$X'' = kX \quad , \; X(0) = 0, \; X(1) = 0$$

$$X = A\cos\sqrt{|k|}\,x + B\sin\sqrt{|k|}\,x$$

$X(0) = A = 0$ so $A = 0$

$X(1) = B\sin\sqrt{|k|} = 0$ so $\sqrt{|k|} = n\pi$ where $n = 1, 2, 3 \ldots$

Let $v(x,t)$ be written

$$v(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin n\pi x$$

Now at $t=0$ we have

$$\sum_{n=1}^{\infty} b_n(0)\sin n\pi x = x(e^{-x} - e^{-1})$$

Therefore

$$b_n(0) = 2\int_0^1 x(e^{-x} - e^{-1})\sin n\pi x \, dx$$

$$= \frac{4n\pi}{(1+n^2\pi^2)^2} + 2\frac{(-1)^n}{e}\frac{1-n^2\pi^2}{n\pi(1+n^2\pi^2)^2}$$

Plugging into the differential equation yields

$$\sum_{n=1}^{\infty} b_n'(t)\sin n\pi x = \sum_{n=1}^{\infty} -n^2\pi^2 b_n(t)\sin n\pi x + \frac{2\pi x}{e}\sin 2\pi t$$

Expanding in terms of series

$$\frac{2\pi x}{e}\sin 2\pi t = \frac{2\pi}{e}\sin 2\pi t \sum_{n=1}^{\infty} f_n \sin n\pi x$$

where

$$f_n = 2\int_0^1 x\sin n\pi x\,dx = -\frac{2(-1)^n}{n\pi}$$

Equating coefficients

$$b_n' + n^2\pi^2 b_n = -\frac{4(-1)^n}{ne}\sin 2\pi t$$

$$(b_n e^{n^2\pi^2 t})' = \frac{-4(-1)^n}{ne}\sin 2\pi t\, e^{n^2\pi^2 t}$$

$$b_n e^{n^2\pi^2 t} - b_n(0) = -\int_0^t \frac{4(-1)^n}{ne}\sin 2\pi s\, e^{n^2\pi^2 s}\,ds$$

$$= \frac{4(-1)^n}{ne}\cdot\left(\frac{-2 + 2e^{n^2\pi^2 t}\cos 2\pi t - n^2\pi e^{n^2\pi^2 t}\sin 2\pi t}{\pi(n^4\pi^2+4)}\right)$$

$$= \frac{4(-1)^n}{ne}\cdot\frac{-2 + e^{n^2\pi^2 t}(2\cos 2\pi t - n^2\pi\sin 2\pi t)}{\pi(n^4\pi^2+4)}$$

$$b_n = b_n(0)e^{-n^2\pi^2 t} + \frac{4(-1)^n}{ne}\cdot\frac{2\cos 2\pi t - n^2\pi\sin 2\pi t - 2e^{-n^2\pi^2 t}}{\pi(n^4\pi^2+4)}$$

$$= \left[\frac{4n\pi}{(1+n^2\pi^2)^2} + 2\frac{(-1)^n}{e}\frac{1-n^2\pi^2}{n\pi(1+n^2\pi^2)^2}\right]e^{-n^2\pi^2 t}$$

$$+ \frac{4(-1)^n}{ne}\cdot\frac{2\cos 2\pi t - n^2\pi\sin 2\pi t - 2e^{-n^2\pi^2 t}}{\pi(n^4\pi^2+4)}$$

It follows that

$$v(x,t) = \sum_{n=1}^{\infty}\left(\frac{4n\pi}{(1+n^2\pi^2)^2} + 2\,\frac{(-1)^n}{e}\,\frac{1-n^2\pi^2}{n\pi(1+n^2\pi^2)^2}\right)e^{-n^2\pi^2 t}\sin n\pi x$$

$$+ \frac{4(-1)^n}{ne}\cdot\frac{2\cos 2\pi t - n^2\pi\sin 2\pi t - 2e^{-n^2\pi^2 t}}{\pi(n^4\pi^2+4)}\sin n\pi x$$

and

$$u(x,t) = \frac{x}{e}\cos 2\pi t + v(x,t)$$

$$= \frac{x}{e}\cos 2\pi t + \sum_{n=1}^{\infty}\left(\frac{4n\pi}{(1+n^2\pi^2)^2} + 2\,\frac{(-1)^n}{e}\,\frac{1-n^2\pi^2}{n\pi(1+n^2\pi^2)^2}\right)e^{-n^2\pi^2 t}\sin n\pi x$$

$$+ \sum_{n=1}^{\infty}\frac{4(-1)^n}{ne}\cdot\frac{2\cos 2\pi t - n^2\pi\sin 2\pi t - 2e^{-n^2\pi^2 t}}{\pi(n^4\pi^2+4)}\sin n\pi x$$

#2 $u_{tt} + 9u_t = u_{xx}$

$u(0,t)=0$

$u(2,t)=0$

$u(x,0)= \sin(2\pi x)$

$u_t(x,0) = \sin(8\pi x)$

Seperation of variables

$$XT'' + 9XT' = X''T$$

$$\frac{T''+9T'}{T} = \frac{X''}{X} = k$$

solve $X'' = kX$, $X(0)=0$, $X(2)=0$

case $k<0$, $X = A\cos\sqrt{|k|}x + B\sin\sqrt{|k|}x$

$X(0) = A = 0$, $A=0$

$X(2) = B\sin 2\sqrt{|k|}$ $\quad 2\sqrt{|k|} = n\pi$, $n=1,2,\dots$

case $k \geq 0$ no non-trivial solutions.

$$T'' + 9T' = -\frac{n^2\pi^2}{4}T$$

$$T'' + 9T' + \frac{n^2\pi^2}{4}T = 0$$

characteristic equation is

$$r^2 + 9r + \frac{n^2\pi^2}{4} = 0$$

$$r = \frac{-9 \pm \sqrt{9^2 - 4\cdot\frac{n^2\pi^2}{4}}}{2} = -\frac{9}{2} \pm \frac{1}{2}\sqrt{81 - n^2\pi^2}$$

Case $n > \frac{9}{\pi}$, then $n^2\pi^2 > 81$ and solutions are of the form

$$T_n = e^{-\frac{9}{2}t}\left(A\cos\frac{t}{2}\sqrt{n^2\pi^2-81} + B\sin\frac{t}{2}\sqrt{n^2\pi^2-81}\right)$$

Case $n \leq \frac{9}{\pi}$ then

$$T_n = A e^{-\frac{t}{2}\left(9+\sqrt{81-n^2\pi^2}\right)} + B e^{-\frac{t}{2}\left(9-\sqrt{81-n^2\pi^2}\right)}$$

In particular the initial conditions

$$u(x,0) = \sin 2\pi x = \sin \frac{4\pi x}{2}$$

$$u_t(x,0) = \sin 3\pi x = \sin \frac{6\pi x}{2}$$

implies that

$$T_n(0) = \begin{cases} 1 & \text{if } n=4 \\ 0 & \text{otherwise} \end{cases}$$

$$T_n'(0) = \begin{cases} 1 & \text{if } n=6 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $T_n(x) = 0$ for $n \neq 4$ and $n \neq 6$. Moreover as both 4 and 6 are greater than $9/\pi$ then

$$T_4 = e^{-\frac{9}{2}t}\left(A_4 \cos \tfrac{t}{2}\sqrt{16\pi^2-81} + B_4 \sin \tfrac{t}{2}\sqrt{16\pi^2-81}\right)$$

$$T_4(0) = A_4 = 1 \qquad \text{implies } A_4 = 1$$

$$T_4' = -\frac{9}{2}T_4 + \frac{1}{2}\sqrt{\;\;}\; e^{-\frac{9}{2}t}\left(-A_4 \sin \tfrac{t}{2}\sqrt{\;\;} + B_4 \cos \tfrac{t}{2}\sqrt{\;\;}\right)$$

$$T_4'(0) = -\frac{9}{2} + \frac{1}{2}\sqrt{\;\;}\; B_4 = 0$$

$$B_4 = \frac{9}{\sqrt{\;\;}} = \frac{9}{\sqrt{16\pi^2-81}}$$

$$T_4 = e^{-\frac{9}{2}t}\left(\cos \tfrac{t}{2}\sqrt{16\pi^2-81} + \frac{9}{\sqrt{16\pi^2-81}} \sin \tfrac{t}{2}\sqrt{16\pi^2-81}\right)$$

Similarly

$$T_6 = e^{-\frac{9}{2}t}\left(A_6 \cos \tfrac{t}{2}\sqrt{36\pi^2-81} + B_6 \sin \tfrac{t}{2}\sqrt{36\pi^2-81}\right)$$

$$T_6(0) = A_6 = 0$$

$$T_6' = -\frac{9}{2}T_6 + \frac{1}{2}\sqrt{\;\;}\; e^{-\frac{9}{2}t} B_6 \cos \tfrac{t}{2}\sqrt{\;\;}$$

$$T_6'(0) = \frac{1}{2}\sqrt{\;\;}\; B_6 = 1 \qquad B_6 = \frac{2}{\sqrt{36\pi^2-81}}$$

$$T_6 = e^{-\frac{9}{2}t} \frac{2}{\sqrt{36\pi^2-81}} \sin \tfrac{t}{2}\sqrt{36\pi^2-81}$$

Now,

$$u(x,t) = T_4 X_4 + T_6 X_6$$

$$= e^{-\frac{9}{2}t}\left(\cos\frac{t}{2}\sqrt{16\pi^2-81} + \frac{9}{\sqrt{16\pi^2-81}}\sin\frac{t}{2}\sqrt{16\pi^2-81}\right)\sin 2\pi x$$

$$+ e^{-\frac{9}{2}t}\frac{2}{\sqrt{36\pi^2-81}}\sin\frac{t}{2}\sqrt{36\pi^2-81}\ \sin 3\pi x$$

#3 $2u_x + x u_y = \sqrt{u}$

$u(0,y) = 1+\sin y$

$\frac{dx}{dt} = 2 \qquad x = 2t$

$\frac{dy}{dt} = x = 2t \qquad y = t^2 + C$

$\frac{du}{dt} = \sqrt{u} \qquad \int \frac{du}{\sqrt{u}} = \int dt\,, \qquad 2\sqrt{u} = t + D$

Therefore

$u(2t, t^2+C) = \frac{1}{4}(t+D)^2$

Let $t=0$ then

$u(0,C) = 1+\sin C = \frac{1}{4}D^2 \qquad \frac{1}{2}D = \pm\sqrt{1+\sin C}$

$D = \pm 2\sqrt{1+\sin C}$

Therefore

$u(2t, t^2+C) = \frac{1}{4}\left(t \pm 2\sqrt{1+\sin C}\right)^2$

$2t = x \qquad t = \frac{x}{2} \qquad t^2 + C = y \quad \frac{x^2}{4} + C = y$

so $C = y - x^2/4$

Hence

$u(x,y) = \frac{1}{4}\left(\frac{x}{2} \pm 2\sqrt{1+\sin(y - x^2/4)}\right)^2$

we choose + solution and note that

$u(x,y) = \frac{1}{4}\left(\frac{x}{2} + 2\sqrt{1+\sin(y-x^2/4)}\right)^2$

is a solution only when $\frac{x}{2} > -2\sqrt{1+\sin(y-x^2/4)}$.

```
> restart;
```

```
> #Maple worksheet to check #3 on homework #3
  #Math 488/688 Spring 2010 by Eric Olson
```

```
> u:=1/4*(x/2+2*sqrt(1+sin(y-x^2/4)))^2;
```

$$u := \frac{1}{4}\left(\frac{1}{2}\,x + 2\,\sqrt{1 - \sin\left(-y + \frac{1}{4}\,x^2\right)}\right)^2$$

```
> ux:=diff(u,x);
  uy:=diff(u,y);
```

$$ux := \frac{1}{2}\left(\frac{1}{2}\,x + 2\,\sqrt{1 - \sin\left(-y + \frac{1}{4}\,x^2\right)}\right)\left(\frac{1}{2} - \frac{\cos\left(-y + \frac{1}{4}\,x^2\right)\,x}{2\,\sqrt{1 - \sin\left(-y + \frac{1}{4}\,x^2\right)}}\right)$$

$$uy := \frac{\left(\frac{1}{2}\,x + 2\,\sqrt{1 - \sin\left(-y + \frac{1}{4}\,x^2\right)}\right)\cos\left(-y + \frac{1}{4}\,x^2\right)}{2\,\sqrt{1 - \sin\left(-y + \frac{1}{4}\,x^2\right)}}$$

```
> A1:=2*ux+x*uy;
```

$$A1 := \left(\frac{1}{2}\,x + 2\,\sqrt{1 - \sin\left(-y + \frac{1}{4}\,x^2\right)}\right)\left(\frac{1}{2} - \frac{\cos\left(-y + \frac{1}{4}\,x^2\right)\,x}{2\,\sqrt{1 - \sin\left(-y + \frac{1}{4}\,x^2\right)}}\right) + \frac{x\left(\frac{1}{2}\,x + 2\,\sqrt{1 - \sin\left(-y + \frac{1}{4}\,x^2\right)}\right)\cos\left(-y + \frac{1}{4}\,x^2\right)}{2\,\sqrt{1 - \sin\left(-y + \frac{1}{4}\,x^2\right)}}$$

```
> A2:=simplify(A1);
```

$$A2 := \frac{1}{4}\,x + \sqrt{1 - \sin\left(-y + \frac{1}{4}\,x^2\right)}$$

```
> A3:=simplify(sqrt(u)) assuming x::real, y::real;
```

$$A3 := \frac{1}{4}\left|x + 4\sqrt{1 + \sin\left(y - \frac{1}{4}x^2\right)}\right|$$

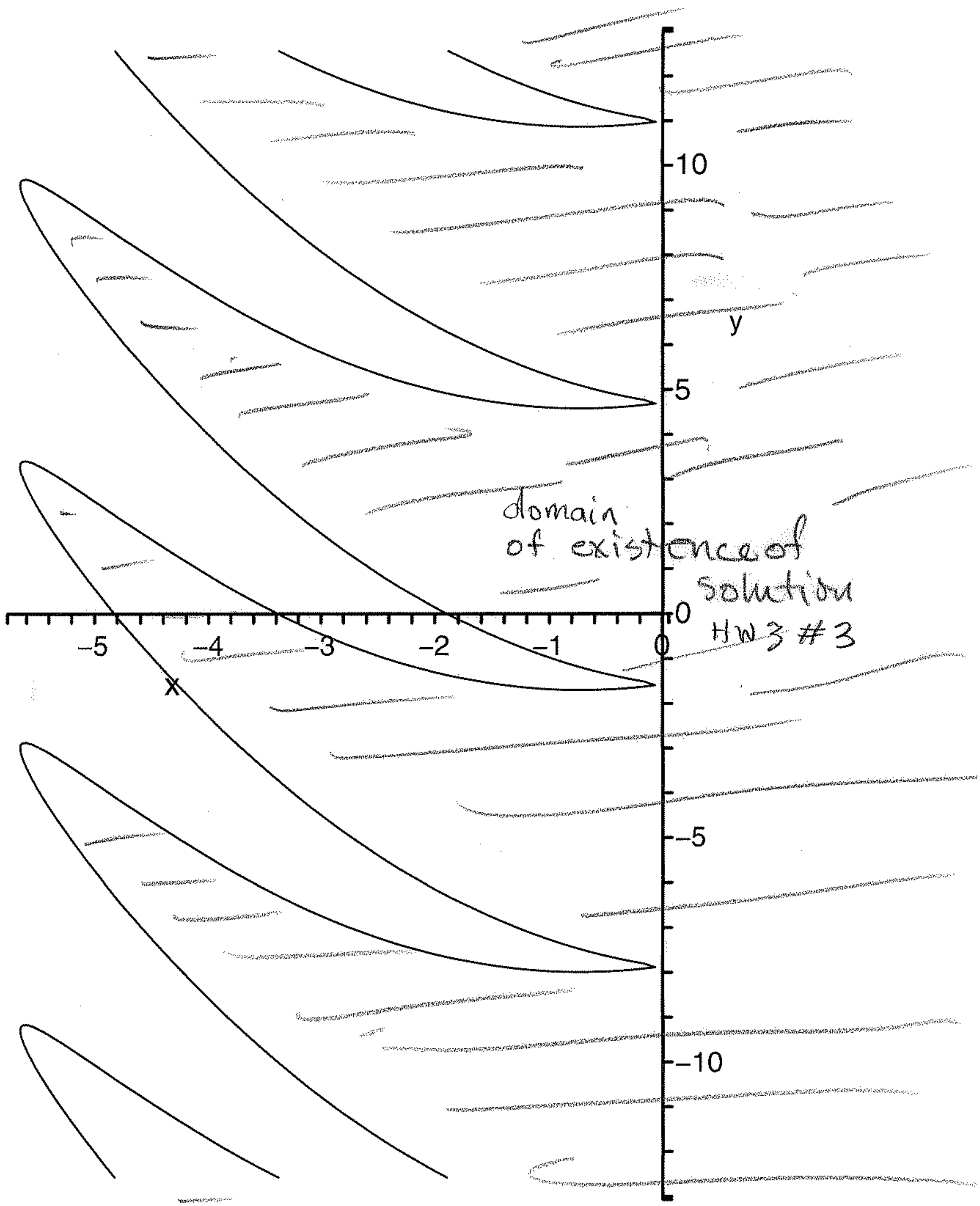
```
> A2-A3;
```

$$\frac{1}{4}x + \sqrt{1 - \sin\left(-y + \frac{1}{4}x^2\right)} - \frac{1}{4}\left|x + 4\sqrt{1 - \sin\left(-y + \frac{1}{4}x^2\right)}\right|$$

```
> #this is zero only when what is in the absolute values is positive
> eq:=x+4*sqrt(1+sin(y-x^2/4));
```

$$eq := x + 4\sqrt{1 - \sin\left(-y + \frac{1}{4}x^2\right)}$$

```
> with(plots):
> contourplot(eq,x=-4*Pi..0*Pi,y=-4*Pi..4*Pi,numpoints=100000,contours=[0])
```



y
10
5
0
–5
–10
x
–5
–4
–3
–2
–1
0
domain of existence of solution
HW3 #3

#4 $u_t = 4u_{xx}$ for $-\infty < x < \infty$

$u(x,0) = x e^{-x^2}$

Using formula 4.3.13 from Evans, Blackledge and Yardley

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + 2cw\sqrt{t})\, e^{-w^2} dw$$

where

$$f(x) = x e^{-x^2} \quad \text{and} \quad c = 2$$

we obtain

$$u(x,t) = \frac{x e^{-x^2/(16t+1)}}{(16t+1)^{3/2}}$$

See attached Maple worksheet for computation of the integral.

```
> restart;
```
#4 calculation...

```
> u:=1/sqrt(Pi)*int(f(x+2*c*w*sqrt(t))*exp(-w^2),w=-infinity..infinity);
```

$$u := \frac{\int_{-\infty}^{\infty} f(x + 2\, c\, w \sqrt{t})\, \mathrm{e}^{(-w^2)}\, dw}{\sqrt{\pi}}$$

```
> f:=x->x*exp(-x^2);
```

$$f := x \to x\, \mathrm{e}^{(-x^2)}$$

```
> c:=2;
```

$$c := 2$$

```
> u assuming t::positive;
```

$$\frac{\mathrm{e}^{\left(\frac{x^2}{-16\, t - 1}\right)}\, x}{(16\, t + 1)^{(3/2)}}$$

```
>
```

#5 Show that the bounded solution $u(x,y)$ of Laplace's equation $\nabla^2 u = 0$ in the quadrant $x>0$, $y>0$ which satisfies $u(0,y)=0$ for $y>0$ and $u(x,0)=f(x)$ for $x>0$ is

$$u(x,y) = \frac{2}{\pi}\int_0^\infty\int_0^\infty e^{-\lambda y} f(\xi)\sin\lambda x \sin\lambda\xi \, d\xi\, d\lambda .$$

By integrating with respect to λ transform this solution into

$$u(x,y) = \frac{y}{\pi}\int_0^\infty f(\xi)\left[\frac{1}{y^2+(\xi-x)^2} - \frac{1}{y^2+(\xi+x)^2}\right] d\xi .$$

By separation of variables

$$X''Y + Y''X = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = k$$

Thus $X'' = kX$, $X(0)=0$. Bounded solutions are when $k<0$.
Therefore

$$X = A\cos\sqrt{|k|}\,x + B\sin\sqrt{|k|}\,x$$

$$X(0) = A = 0 \quad , \quad A = 0.$$

Hence $X = B\sin\sqrt{|k|}\,x$. Introducing the parameter $\lambda^2 = -k$ we obtain

$$X_\lambda = B_\lambda \sin\lambda x .$$

Now write

$$u(x,y) = \int_0^\infty b_\lambda(y)\sin\lambda x\, d\lambda .$$

Then $u(x,0) = f(x)$ implies

$$\int_0^\infty b_\lambda(0)\sin\lambda x\, d\lambda = f(x)$$

It follow from the Fourier sine inversion formula 4.4.2 that

$$b_\lambda(0) = \frac{2}{\pi}\int_0^\infty f(s)\sin\lambda s\, ds$$

Now plugging $u(x,y)$ into $u_{xx}+u_{yy}=0$ yields

$$\int_0^\infty -\lambda^2 b_\lambda(y)\sin\lambda x\,d\lambda + \int_0^\infty b_\lambda''(y)\sin\lambda x\,d\lambda = 0$$

Since the $\sin\lambda x$ are orthogonal the equating coefficients gives

$$b_\lambda'' - \lambda^2 b_\lambda = 0$$

so $\quad b_\lambda = Ce^{\lambda y} + De^{-\lambda y}.$

In order that the solution is bounded for $y>0$ we obtain that $C=0$. Thus

$$b_\lambda = De^{-\lambda y}.$$

Moreover

$$b_\lambda(0) = D = \frac{2}{\pi}\int_0^\infty f(s)\sin\lambda s\,ds.$$

Therefore

$$b_\lambda = \frac{2}{\pi}e^{-\lambda y}\int_0^\infty f(s)\sin\lambda s\,ds.$$

Substituting into the formula for $u(x,y)$ obtains

$$u(x,y) = \int_0^\infty \left(\frac{2}{\pi}e^{-\lambda y}\int_0^\infty f(s)\sin\lambda s\,ds\right)\sin\lambda x\,d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty\int_0^\infty e^{-\lambda y} f(s)\sin\lambda x\,\sin\lambda s\,ds\,d\lambda$$

which is the formula desired.

Now interchange the order of integration

$$u(x,y) = \frac{2}{\pi}\int_0^\infty f(s)\left(\int_0^\infty e^{-\lambda y}\sin\lambda x\sin\lambda s\,d\lambda\right)ds$$

and integrate the integral in λ

Trigonometry

$$\cos\lambda(x+s) = \cos\lambda x \cos\lambda s - \sin\lambda x \sin\lambda s$$
$$\cos\lambda(x-s) = \cos\lambda x \cos\lambda s + \sin\lambda x \sin\lambda s$$

subtracting

$$\cos\lambda(x-s) - \cos\lambda(x+s) = 2\sin\lambda x \sin\lambda s$$

so

$$2\int_0^\infty e^{-\lambda y} \sin\lambda x \sin\lambda s \, d\lambda$$

$$= \int_0^\infty e^{-\lambda y} \cos\lambda(x-s)\, d\lambda - \int_0^\infty e^{-\lambda y} \cos\lambda(s+s)\, d\lambda$$

$$= \frac{y}{y^2+(x-s)^2} - \frac{y}{y^2+(x+s)^2}$$

Therefore

$$u(x,y) = \frac{1}{\pi}\int_0^\infty f(s)\left(\frac{y}{y^2+(x-s)^2} - \frac{y}{y^2+(x+s)^2}\right) ds$$

$$= \frac{y}{\pi}\int_0^\infty f(s)\left(\frac{1}{y^2+(x-s)^2} - \frac{1}{y^2+(x+s)^2}\right) ds$$