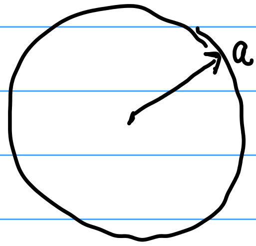
Laplace on a disc.



Still works

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0 \qquad r\in[0,a],\ \theta\in[-\pi,\pi]$$

$u(a,\theta)=f(\theta)$ $\qquad \theta\in[-\pi,\pi]$ $\qquad$ inhomogene

$$\boxed{\begin{aligned} &u(r,-\pi)=u(r,\pi)\\ &u_\theta(r,-\pi)=u_\theta(r,\pi)\end{aligned}} \qquad r\in[0,a] \qquad \text{homogeneo}$$

$u(r,\theta)=\varphi(\theta)\,G(r)$

Obtain 2 ODEs

$$\boxed{\begin{aligned} &\varphi''=-\lambda\varphi\\ &\varphi(-\pi)=\varphi(\pi)\\ &\varphi'(-\pi)=\varphi'(\pi)\end{aligned}} \qquad \text{and} \qquad r\frac{d}{dr}\left(r\frac{dG}{dr}\right)=\lambda G$$

Solve φ:

Case $\lambda>0$: Then $\varphi(\theta)=c_1\cos n\theta+c_2\sin n\theta$

and $\lambda=n^2$, $n=1,2,\dots$

Case $\lambda=0$: Then $\varphi''=0$ so $\varphi(\theta)=c_1\theta+c_2$

Solve for B.C.s $\varphi(-\pi)=c_1(-\pi)+\cancel{c_2}$

$=\varphi(\pi)=c_1(\pi)+\cancel{c_2}$

Thus $2c_1\pi=0$ so $c_1=0$

other $\varphi'(\theta)=c_1$ so $\varphi'(-\pi)=0=\varphi'(\pi)$

no addition restrictions

Thus $\varphi(\theta)=c_2$

Case $\underline{\lambda < 0}$. General solution $\varphi(\theta) = c_1 \cosh\sqrt{\lambda}\,\theta + c_2 \sinh\sqrt{\lambda}\,\theta$,

$$\varphi(-\pi) = c_1 \cancel{\cosh}\sqrt{\lambda}\,\pi - c_2 \sinh\sqrt{\lambda}\,\pi,$$

$$= \varphi(\pi) = c_1 \cancel{\cosh}\sqrt{\lambda}\,\pi + c_2 \sinh\sqrt{\lambda}\,\pi.$$

$$2 c_2 \underbrace{\sinh\sqrt{\lambda}\,\pi}_{\text{not zero so}} = 0 \quad \text{implies} \quad c_2 = 0$$

Also $\varphi'(\theta) = c_1\sqrt{\lambda}\,\sinh\sqrt{\lambda}\,\theta + \cancel{c_2\sqrt{\lambda}\,\cosh\sqrt{\lambda}\,\theta}$

$$\varphi'(-\pi) = -c_1\sqrt{\lambda}\,\sinh\sqrt{\lambda}\,\pi$$

$$= \varphi'(\pi) = c_1\sqrt{\lambda}\,\sinh\sqrt{\lambda}\,\pi \qquad \text{thus } c_1 = 0$$

No non-zero solutions when $\lambda < 0$.

In summary the solutions are

$$\varphi_n(\theta) = c_1 \cos n\theta + c_2 \sin n\theta \qquad \text{for } n = 1, 2, \ldots$$

$$\varphi_0(\theta) = c_2$$

Now solve the ODE for G.

$$r \frac{d}{dr}\left(r \frac{dG}{dr}\right) = \lambda G$$

Recognize this ODE as an Euler equation.

Technique is substitute $G = r^p$ and solve for p.

Thus

$$r \frac{d}{dr}\left(r \frac{d}{dr} r^p\right) = \lambda r^p$$

$$r\frac{d}{dr}\left(rp\,r^{p-1}\right) = \lambda r^p$$

$$r\frac{d}{dr}\left(p\,r^p\right) = \lambda r^p$$

$$r\,p^2\,r^{p-1} = \lambda r^p$$

$$p^2 \cancel{r^p} = \lambda \cancel{r^p} \qquad p^2 = \lambda = n^2$$

Thus $p = \pm n$

General solution $G(r) = c_1 r^n + c_2 r^{-n}$.

for $r \in [0,a]$

Note since the domain of this ODE includes $r=0$ then have to set $c_2 = 0$ so the solution exists and is bounded on the domain.

Thus

$$G_n(r) = c_1 r^n$$

$$\phi_n(\theta) = c_1 \cos n\theta + c_2 \sin n\theta \qquad \text{for } n = 1, 2, \ldots$$

and..., for $n=0$ we have

$$\cancel{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = 0 \qquad \frac{d}{dr}\left(r\frac{dG}{dr}\right) = 0$$

$$r\frac{dG}{dr} = c_1 \qquad \frac{dG}{dr} = \frac{c_1}{r}$$

$$G(r) = c_1 \log r + c_2 \qquad \text{for } r \in [0,a]$$

Note since the domain of this ODE includes $r=0$ then have to set $c_1=0$ so the solution exists and is bounded on the domain.

Thus $G_0(r) = c_2$

$\varphi_0(\vartheta) = c_2$

product of these constants is A_0

Superposition Gives that

$$u(r,\theta) = \varphi_0(\theta)G_0(r) + \sum_{n=1}^{\infty} \varphi_n(\theta)G_n(r)$$

$$= A_0 + \sum_{n=1}^{\infty} \left(A_n(\cos n\theta) + B_n(\sin n\vartheta)\right) r^n$$

Now solve for the A's and B's using the inhomogeneous boundary condition $u(a,\theta)=f(\theta)$

$$u(a,\theta) = A_0 + \sum_{n=1}^{\infty} \left(A_n(\cos n\theta) + B_n(\sin n\vartheta)\right) a^n = f(\theta).$$

mult by either $\cos m\vartheta$ or $\sin n\vartheta$ and integrate for $\theta \in [-\pi,\pi]$ and use orthogonality.

First only integrate over $[-\pi,\pi]$ to find A_0

$$\int_{-\pi}^{\pi} \left(A_0 + \sum_{n=1}^{\infty} \left(\cancel{A_n(\cos n\theta)} + \cancel{B_n(\sin n\vartheta)}\right) a^n\right) d\theta = \int_{-\pi}^{\pi} f(\theta)\,d\vartheta$$

$$2\pi A_0 = \int_{-\pi}^{\pi} f(\theta)\,d\theta \qquad A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

$$\cos m\theta \left[A_0 + \sum_{n=1}^{\infty} \left(A_n(\cos n\theta) + B_n(\sin n\theta)\right) a^n\right] = \cos m\theta\, f(\theta).$$

$$\int_{-\pi}^{\pi} \cos m\theta \left[A_0 + \sum_{n=1}^{\infty} \left(A_n(\cos n\theta) + B_n(\sin n\theta)\right) a^n\right] d\theta = \int_{-\pi}^{\pi} \cos m\theta\, f(\theta)\, d\theta$$

$$\frac{2\pi}{2} A_m a^m = \int_{-\pi}^{\pi} \cos m\theta\, f(\theta)\, d\theta$$

$$A_m = \frac{1}{\pi a^m} \int_{-\pi}^{\pi} \cos m\theta\, f(\theta)\, d\theta$$

Similarly

$$B_m = \frac{1}{\pi a^m} \int_{-\pi}^{\pi} \sin m\theta\, f(\theta)\, d\theta$$

In summary the solution to

still works

PDE $\quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad r \in [0,a],\ \theta \in [-\pi,\pi]$

BC. $\quad u(a,\theta) = f(\theta) \qquad \theta \in [-\pi,\pi]$ inhomogeneous

$\quad \boxed{u(r,-\pi) = u(r,\pi)} \qquad r \in [0,a]$ homogeneous

$\quad u_\theta(r,-\pi) = u_\theta(r,\pi)$

is

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} \left(A_n(\cos n\theta) + B_n(\sin n\theta)\right) r^n$$

where

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\vartheta)\, d\vartheta$$

and

$$A_m = \frac{1}{\pi a^m}\int_{-\pi}^{\pi} \cos m\vartheta \; f(\vartheta)\, d\vartheta \quad \text{and} \quad B_m = \frac{1}{\pi a^m}\int_{-\pi}^{\pi} \sin m\vartheta \; f(\vartheta)\, d\vartheta$$