Fourier series $g(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}.$

Converges pointwise provided: g is piecewise smooth.

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.$$

$$g'(x) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} \beta_n \sin\frac{n\pi x}{L}$$

$$\alpha_0 = \frac{1}{2L}\int_{-L}^{L} f'(x)\,dx$$

$$\alpha_n = \frac{1}{L}\int_{-L}^{L} f'(x)\cos\frac{n\pi x}{L}\,dx$$

$$\beta_n = \frac{1}{L}\int_{-L}^{L} f'(x)\sin\frac{n\pi x}{L}\,dx$$

Converges pointwise provided g' is piecewise smooth...,

Note: g' is the periodic extension of f' with x's added at the jump discontinuities at the average value of the jump.

$$\frac{d}{dx} g(x) = \frac{d}{dx}\left(a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\right)$$

$$\beta_n = -\frac{n\pi}{L} a_n$$

If I could differentiate term by term then

$$g'(x) \overset{?}{=} \underbrace{0}_{\alpha_0} + \sum_{n=1}^{\infty} \underbrace{-\frac{n\pi}{L} a_n}_{?\ \beta_n} \sin\frac{n\pi x}{L} + \sum_{n=1}^{\infty} \underbrace{\frac{n\pi}{L} b_n}_{\alpha_n ?} \cos\frac{n\pi x}{L}$$

If $f'(x)$ is piecewise smooth, then the Fourier series of a continuous function $f(x)$ can be differentiated term by term if $f(-L) = f(L)$.

$$\alpha_0 = \frac{1}{2L}\int_{-L}^{L} f'(x)\,dx = \frac{1}{2L} f(x)\Big|_{-L}^{L} = \frac{1}{2L}\left(\cancel{f(L)} - \cancel{f(-L)}\right) = 0$$

since $f(-L) = f(L)$
by hypothesis

need to get the prime
away from the f.. integration by parts...

$$\alpha_n = \frac{1}{L}\int_{-L}^{L} f'(x)\cos\frac{n\pi x}{L}\,dx$$

$u = \cos\frac{n\pi x}{L}$ $\qquad du = -\frac{n\pi}{L}\sin\frac{n\pi x}{L}$

$dv = f'(x)\,dx$ $\qquad v = f(x)$

$$= \frac{1}{L}\left(\cos\frac{n\pi x}{L} f(x)\Big|_{-L}^{L} + \int_{-L}^{L}\frac{n\pi}{L}\sin\frac{n\pi x}{L} f(x)\,dx\right)$$

$$= \frac{1}{L}\left[\left(\cancel{\cos n\pi}\right) f(L) - \cancel{\cos(-n\pi)}\, f(-L)\right] + \frac{1}{L}\int_{-L}^{L}\frac{n\pi}{L}\sin\frac{n\pi x}{L} f(x)\,dx$$

$$= \frac{n\pi}{L}\cdot\frac{1}{L}\int_{-L}^{L}\sin\frac{n\pi x}{L} f(x)\,dx$$

$$\alpha_n = \frac{n\pi}{L} b_n \quad ☑$$

recall

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.$$

$$\beta_n = \frac{1}{L}\int_{-L}^{L} f'(x)\sin\frac{n\pi x}{L}\,dx$$

$$u = \sin\frac{n\pi x}{L} \qquad du = \frac{n\pi}{L}\cos\frac{n\pi x}{L}$$

$$dv = f'(x)\,dx \qquad v = f(x)$$

$$= \frac{1}{L}\left(\sin\frac{n\pi x}{L} f(x)\Big|_{-L}^{L} - \int_{-L}^{L} \frac{n\pi}{L}\cos\frac{n\pi x}{L} f(x)\,dx \right)$$

$$= \frac{1}{L}\Big[(\sin n\pi) f(L) - \sin(-n\pi) f(-L) \Big] - \frac{1}{L}\int_{-L}^{L} \frac{n\pi}{L}\cos\frac{n\pi x}{L} f(x)\,dx$$

here the fact that $\sin(n\pi)=0$ is used

$$= \frac{n\pi}{L}\underbrace{-\frac{1}{L}\int_{-L}^{L}\cos\frac{n\pi x}{L} f(x)\,dx} = -\frac{n\pi}{L}a_n$$

Thus

$$\beta_n = -\frac{n\pi}{L}a_n$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

Thus the differentiation of the Fourier series with respect to x depends on integration by parts and we need to check the hypothesis of the integration by parts formula to really have a proof.

Recall uv apply product rule

$$(uv)' = u'v + uv'$$

$$\underbrace{\int (uv)'\,dx}_{\text{fund theorem of calculus}} = \int u'v\,dx + \int uv'\,dx$$

$$uv = \int u'v\,dx + \int uv'\,dx$$

$$\int_a^b uv'\,dx = uv\Big|_a^b - \int_a^b u'v\,dx$$

In our case we are working with

$$\int_{-L}^{L} f'(x) \underbrace{\sin \frac{n\pi x}{L}}_{\text{this function is very smooth}} dx$$

only problems here...

In this case integration by parts works even when f' is only piecewise smooth...

Validity for less smooth functions [edit]

It is not necessary for u and v to be continuously differentiable. Integration by parts works if u is absolutely continuous and the function designated v' is Lebesgue integrable (but not necessarily continuous).[3] (If v' has a point of discontinuity then its antiderivative v may not have a derivative at that point.)

If the interval of integration is not compact, then it is not necessary for u to be absolutely continuous in the whole interval or for v' to be Lebesgue integrable in the interval, as a couple of examples (in which u and v are continuous and continuously differentiable) will show. For instance, if

Since sine and cosine are so nice the formula works for us...

Superposition of solutions

$$u(x,t) = \sum_n a_n \phi_n(x)\, G_n(t)$$

$$= \sum_n a_n G_n(t) \sin\frac{n\pi x}{L}$$

We've already discussed differentiating this series with respect to x. But what about t?

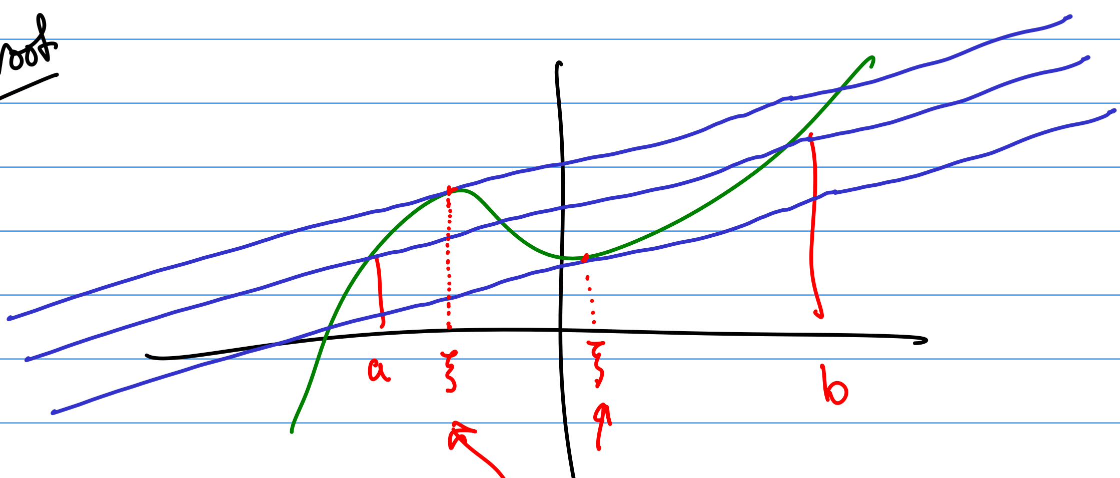
$$\frac{\partial u}{\partial t} \stackrel{?}{=} \sum_n a_n G_n'(t) \sin\frac{n\pi x}{L}$$

can switch the limits?

Mean value theorem. If f is continuously differentiable on $[a,b]$ then there is $\xi \in [a,b]$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(\xi)$$

Proof



two different choices for ξ.

Consider the Fourier series

$$g(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{L}$$

$$\frac{\partial}{\partial t} g(x,t) = \lim_{h \to 0} \frac{g(x,t+h) - g(x,t)}{h}$$

$$= \lim_{h \to 0} \sum_{n=1}^{\infty} \frac{a_n(t+h) - a_n(t)}{h} \sin \frac{n\pi x}{L}$$

$$= \lim_{h \to 0} \sum_{n=1}^{\infty} a_n'(t + \xi(t)) \quad \sin \frac{n\pi x}{L}$$

where ξ is between 0 and h

and depends on t. . .