$$\boxed{g(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\frac{n\pi x}{L}}$$

$$\frac{\partial g(x,t)}{\partial t} = \lim_{h\to 0} \frac{g(x,t+h) - g(x,t)}{h}$$

$$= \lim_{h\to 0} \sum_{n=1}^{\infty} \frac{a_n(t+h) - a_n(t)}{h} \sin\frac{n\pi x}{L}$$

By the mean value theorem

$$\frac{a_n(t+h) - a_n(t)}{h} = a_n'(c)$$

for some c between t and $t+h$.

equivalently

$$\frac{a_n(t+h) - a_n(t)}{h} = a_n'(t+\xi)$$

for some ξ between 0 and h.

Note ξ depends on t

$$\frac{\partial g}{\partial t}(x,t) = \lim_{h\to 0} \sum_{n=1}^{\infty} a_n'(t+\xi(t)) \sin\frac{n\pi x}{L}$$

Assume that ? This isn't going to work because the $\xi(t)$ also depends on n so still can't interchange the limiting process...

Assume g is smooth then

$$g(x,t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t) \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n(t) \sin\frac{n\pi x}{L}$$

Assume that g_t is smooth. So g_{tt} exists and is continuous. Also its Fourier series converges pointwise.
Thus

$$g_t(x,t) = \alpha_0(t) + \sum_{n=1}^{\infty} \alpha_n(t) \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} \beta_n(t) \sin\frac{n\pi x}{L}$$

where

$$\alpha_0(t) = \frac{1}{2L}\int_{-L}^{L} g_t(x,t)\,dx \qquad \alpha_n = \frac{1}{L}\int_{-L}^{L} g_t(x,t) \cos\frac{n\pi x}{L}\,dx$$

$$\beta_n = \frac{1}{L}\int_{-L}^{L} g_t(x,t) \sin\frac{n\pi x}{L}\,dx$$

Goal find a relationship between the as, b's and αs and βs.

If term by term differentiation holds then we would have $\underbrace{\alpha_n(t) = a_n'(t) \quad \text{and} \quad \beta_n(t) = b_n'(t)}_{\text{trying to show this...}}$.

Showing this for α's is similar to βs so let's do one one then

$$\alpha_n(t) = \frac{1}{L}\int_{-L}^{L} g_t(x,t) \cos\frac{n\pi x}{L}\,dx$$

$$a_n(t) = \frac{1}{L}\int_{-L}^{L} g(x,t) \cos\frac{n\pi x}{L}\,dx.$$

$$\frac{d}{dt} a_n(t) = \frac{d}{dt}\left(\frac{1}{L}\int_{-L}^{L} g(x,t)\cos\frac{n\pi x}{L}\,dx\right)$$

$$= \lim_{h\to 0} \frac{\frac{1}{L}\int_{-L}^{L} g(x,t+h)\cos\frac{n\pi x}{L}\,dx - \frac{1}{L}\int_{-L}^{L} g(x,t)\cos\frac{n\pi x}{L}\,dx}{h}$$

$$= \lim_{h\to 0} \frac{1}{L}\int_{-L}^{L} \frac{g(x,t+h)-g(x,t)}{h}\cos\frac{n\pi x}{L}\,dx$$

By the mean value theorem

$$\frac{g(x,t+h)-g(x,t)}{h} = g_t(x,c)$$

for some c between $\boxed{t \text{ and } t+h}$

equivalently

note ξ depends on x and t

$$\frac{g(x,t+h)-g(x,t)}{h} = g_t(x,t+\xi)$$

for some $\boxed{\xi \text{ between } 0 \text{ and } h}$.

$$a_n'(t) = \lim_{h\to 0} \frac{1}{L}\int_{-L}^{L} g_t(x,t+\xi(x,t))\cos\frac{n\pi x}{L}\,dx$$

Note I've assumed g and g_t are smooth. Thus g_{tt} exists and is continuous...

Since g_{tt} is continuous on $[-L,L]$ then it has a minimum and maximum.

This means there is a bound B such that

$$|g_{tt}(x,s)| \le B \quad \text{for all } x \in [-L, L]$$

and s between t and $t+h$

Use the mean value theorem one more time...

cleared the denominator to handle the case $\xi = 0$

$$g_t(x, t+\xi(x,t)) - g_t(x,t) = \xi(x,t)\, g_{tt}(x, c)$$

for some c between t and $t+\xi(x,t)$

equivalently

$$g_t(x, t+\xi(x,t)) - g_t(x,t) = \xi(x,t)\, g_{tt}(x, t+\psi)$$

for some ψ between 0 and $\xi(x,t)$

Note $\psi = \psi(x,t)$.

Since ξ between 0 and h. Then ψ is between 0 and h

$$g_t(x, t+\xi(x,t)) = g_t(x,t) + \xi(x,t)\, g_{tt}(x, t+\psi)$$

$$a_n'(t) = \lim_{h\to 0} \frac{1}{L}\int_{-L}^{L} g_t(x, t + \xi(x,t)) \cos\frac{n\pi x}{L}\,dx$$

this is the dependence on h

$$= \lim_{h\to 0} \frac{1}{L}\int_{-L}^{L} \left(g_t(x,t) + \xi(x,t)\, g_{tt}(x, t+\psi)\right) \cos\frac{n\pi x}{L}\,dx$$

doesn't depend on h

$$= \boxed{\frac{1}{L}\int_{-L}^{L} g_t(x,t) \cos\frac{n\pi x}{L}\,dx} + \lim_{h\to 0} \frac{1}{L}\int_{-L}^{L} \xi(x,t)\, g_{tt}(x, t+\psi) \cos\frac{n\pi x}{L}\,dx$$

recall

$$\alpha_n(t) = \frac{1}{L}\int_{-L}^{L} g_t(x,t) \cos\frac{n\pi x}{L}\,dx$$

$$= \alpha_n(t) + \underbrace{\lim_{h\to 0} \frac{1}{L}\int_{-L}^{L} \xi(x,t)\, g_{tt}(x, t+\psi) \cos\frac{n\pi x}{L}\,dx}_{\text{what's left is to show this limit is zero}}$$

Estimate this:

$$\left|\frac{1}{L}\int_{-L}^{L} \xi(x,t)\, g_{tt}(x, t+\psi) \cos\frac{n\pi x}{L}\,dx\right|$$

ξ between 0 and h

$$\leq \frac{1}{L}\int_{-L}^{L} |\xi(x,t)|\,|g_{tt}(x, t+\psi)|\,\left|\cos\frac{n\pi x}{L}\right|dx$$

Since $\left|\cos\frac{n\pi x}{L}\right| \leq 1$ $\quad |\xi(x,t)| \leq |h|$ $\quad |g_{tt}(x,s)| \leq B$

Then

$$\left| \frac{1}{L} \int_{-L}^{L} \xi(x,t)\, g_t(x, t+\psi) \cos \frac{n\pi x}{L}\, dx \right|$$

$$\leq \frac{1}{L} \int_{-L}^{L} |h|\, B \cdot 1 \, dx = 2hB \to 0$$

as $h \to 0$

Thus

$$a_n'(t) = \alpha_n(t) + \lim_{h \to 0} \frac{1}{L} \int_{-L}^{L} \xi(x,t)\, g_t(x, t+\psi) \cos \frac{n\pi x}{L}\, dx$$

$$= \alpha_n(t)$$

Similarly $b_n'(t) = \beta_n(t)$ ...