4.4.3. Consider a slightly damped vibrating string that satisfies

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}.$$

(a) Briefly explain why $\beta > 0$.

*(b) Determine the solution (by separation of variables) that satisfies the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0$$

and the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

You can assume that this frictional coefficient β is relatively small ($\beta^2 < 4\pi^2 \rho_0 T_0 / L^2$).

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}$$

take energy away from the system..

$$\text{Kinetic energy} = \int_0^L \rho_0 (u_t)^2 \, dx$$

$$\frac{d}{dt}(\text{Kinetic energy}) = \frac{d}{dt} \int_0^L \rho_0 (u_t)^2 \, dx$$

$$= \rho_0 \int_0^L \frac{\partial}{\partial t}(u_t)^2 \, dx = \rho_0 \int_0^L 2 u_t u_{tt} \, dx$$

$$= \int_0^L 2 u_t \left(T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t} \right) dx$$

$$= 2T_0 \int_0^L u_t u_{xx} \, dx - 2\beta \int_0^L (u_t)^2 \, dx$$

Now integrate by parts

$$\int_0^L u_t u_{xx} \, dx = u_x u_t \Big|_0^L - \int_0^L u_x u_{tx} \, dx$$

$du = u_{xx} \, dx$ $\qquad u = u_x$

$v = u_t$ $\qquad dv = u_{tx} \, dx$

where

$$u_x u_t \Big|_0^L = u_x(L)\, u_t(L) - u_x(0)\, u_t(0) = 0$$

recall

$u(0,t) = 0$ and $u(L,t) = 0$

$u_t(0,t) = 0$ $u_t(L,t) = 0$

for all t

Since $u_x u_{tx} = \frac{1}{2}\frac{\partial}{\partial t}(u_x)^2$, Then

$$\int_0^L u_x u_{tx}\, dx = \int_0^L \frac{1}{2}\frac{\partial}{\partial t}(u_x)^2\, dx = \frac{1}{2}\frac{d}{dt}\int_0^L (u_x)^2\, dx$$

Therefore...

$$\frac{d}{dt}\int_0^L \rho_0 (u_t)^2\, dx = \cancel{2}T_0\left(-\frac{1}{\cancel{2}}\frac{d}{dt}\int_0^L (u_x)^2\, dx\right) - 2\beta\int_0^L (u_t)^2\, dx$$

$$\frac{d}{dt}\left(\int_0^L \rho_0 (u_t)^2\, dx + \int_0^L T_0 (u_x)^2\, dx\right) = -2\beta\int_0^L (u_t)^2\, dx$$

$E =$ Kinetic energy $+$ Potential energy.

$$\frac{d}{dt}E = -2\beta\int_0^L (u_t)^2\, dx$$

positive so integral is positive

$\beta > 0$

So the negative sign here means total energy decreases over time since $\beta > 0$.

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You can assume that this frictional coefficient β is relatively small ($\beta^2 < 4\pi^2 \rho_0 T_0/L^2$).

$\beta^2 < 4\pi^2 \rho_0 T_0 / L^2$

Let $\quad u(x,t) = \phi(x)h(t) \quad$ and plug it in...

$$\rho_0 \phi(x) h''(t) = T_0 \phi''(x) h(t) - \beta \phi(x) h'(t)$$

$$\phi(x)\left(\rho_0 h''(t) + \beta h'(t)\right) = T_0 \phi''(x) h(t)$$

$$\frac{\rho_0 h''(t) + \beta h'(t)}{T_0 h(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda$$

Therefore

$$\rho_0 h''(t) + \beta h'(t) = -\lambda T_0 h(t) \quad \text{and} \quad \phi''(x) = -\lambda \phi(x),$$

$$\phi(0) = 0, \; \phi(L) = 0$$

General solution for ϕ:

$$\phi(x) = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x$$

$$\phi(0) = c_1 = 0$$

$$\phi(L) = c_2 \sin\sqrt{\lambda} L = 0 \qquad \sqrt{\lambda} L = n\pi \quad \text{for } n = 1, 2, \ldots$$

$$\lambda = \frac{n^2\pi^2}{L^2}$$

$$\phi(x) = c_2 \sin\frac{n\pi x}{L}$$

Now solve for h: ODE assume $\beta > 0$ and $\beta^2 < 4\pi^2 \rho_0 T_0 / L^2$

$$\rho_0 h''(t) + \beta h'(t) = -\lambda T_0 h(t)$$

$$\rho_0 h''(t) + \beta h'(t) + \lambda T_0 h(t) = 0$$

let $h(t) = e^{rt}$ plug it in.

$$\rho_0 r^2 e^{rt} + \beta r e^{rt} + \lambda T_0 e^{rt} = 0$$

$$\rho_0 r^2 + \beta r + \lambda T_0 = 0$$

$a = \rho_0$ $\quad b = \beta$ $\quad c = \lambda T_0$

recall

$\beta > 0$ and $\beta^2 < 4\pi^2 \rho_0 T_0 / L^2$

$\lambda = \frac{n^2 \pi^2}{L^2}$

Use quadratic formula to solve.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-\beta \pm \sqrt{\beta^2 - 4\rho_0 \lambda T_0}}{2\rho_0} = \frac{-\beta \pm \sqrt{\beta^2 - 4\rho_0 \frac{n^2\pi^2}{L^2} T_0}}{2\rho_0}$$

this is negative since $n^2 \geq 1$ and $\beta^2 < 4\pi^2 \rho_0 T_0 / L^2$

Therefore the general solution is

$$h(t) = e^{-\beta t/2\rho_0}\left(c_1 \cos \frac{\sqrt{4\rho_0 n^2 \pi^2 T_0 / L^2 - \beta^2}}{2\rho_0} t + c_2 \sin \frac{\sqrt{4\rho_0 n^2 \pi^2 T_0 / L^2 - \beta^2}}{2\rho_0} t \right)$$

By the superposition principle

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\beta t/2\rho_0}\left(a_n \cos \frac{\sqrt{4\rho_0 n^2 \pi^2 T_0 / L^2 - \beta^2}}{2\rho_0} t + b_n \sin \frac{\sqrt{4\rho_0 n^2 \pi^2 T_0 / L^2 - \beta^2}}{2\rho_0} t \right) \sin \frac{n\pi x}{L}$$

Now solve for a_n and b_n using the initial conditions...

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

$$u(x,0) = \sum_{n=1}^{\infty} e^{0}\left(a_n \cos 0 + b_n \sin 0\right) \sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} = f(x)$$

$$\boxed{a_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx}$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\sum_{n=1}^{\infty} \underbrace{e^{-\beta t/2\rho_0}\Bigg(a_n}_{\text{product rule}} \cos\frac{\sqrt{4\rho_0 n^2\pi^2 T_0/L^2-\beta^2}}{2\rho_0}t + b_n \sin\frac{\sqrt{4\rho_0 n^2\pi^2 T_0/L^2-\beta^2}}{2\rho_0}t\Bigg)\sin\frac{n\pi x}{L}$$

$$= \sum_{n=1}^{\infty} \frac{-\beta}{2\rho_0} e^{-\beta t/2\rho_0}\left(a_n \cos\frac{\sqrt{4\rho_0 n^2\pi^2 T_0/L^2-\beta^2}}{2\rho_0}t + b_n \sin\frac{\sqrt{4\rho_0 n^2\pi^2 T_0/L^2-\beta^2}}{2\rho_0}t\right)\sin\frac{n\pi x}{L}$$

$$+ \sum_{n=1}^{\infty} \frac{\sqrt{4\rho_0 n^2\pi^2 T_0/L^2-\beta^2}}{2\rho_0} e^{-\beta t/2\rho_0}\left(b_n \cos\frac{\sqrt{4\rho_0 n^2\pi^2 T_0/L^2-\beta^2}}{2\rho_0}t - a_n \sin\frac{\sqrt{4\rho_0 n^2\pi^2 T_0/L^2-\beta^2}}{2\rho_0}t\right)\sin\frac{n\pi x}{L}$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \frac{-\beta}{2\rho_0} a_n \sin\frac{n\pi x}{L} + \sum_{n=1}^{\infty} \frac{\sqrt{4\rho_0 n^2\pi^2 T_0/L^2-\beta^2}}{2\rho_0} b_n \sin\frac{n\pi x}{L} = g(x)$$

Therefore

$$\frac{\sqrt{4\rho_0 n^2\pi^2 T_0/L^2-\beta^2}}{2\rho_0} b_n = \frac{\beta}{2\rho_0} a_n + \frac{2}{L}\int_0^L g(x) \sin\frac{n\pi x}{L}\, dx$$

or

$$b_n = \frac{2\rho_0}{\sqrt{4\rho_0 n^2\pi^2 T_0/L^2-\beta^2}}\left(\frac{\beta}{2\rho_0} a_n + \frac{2}{L}\int_0^L g(x) \sin\frac{n\pi x}{L}\, dx\right)$$