6. Any eigenvalue can be related to its eigenfunction by the **Rayleigh quotient**:

$$\lambda = \frac{-p\phi\, d\phi/dx\big|_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2]\, dx}{\int_a^b \phi^2 \sigma\, dx},$$

where the boundary conditions may somewhat simplify this expression.

$$L\phi = \frac{d}{dx}\left(p(x)\,\phi'(x)\right) + q(x)\,\phi(x)$$

the PDE $\quad L\phi = -\lambda\sigma\phi$

$$\beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) = 0$$

$$\beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) = 0,$$

and $p > 0$ and $\sigma > 0$

Goal to solve for λ

Linear Algebra motivation $\quad A \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$.

$Ax = \lambda x \quad$ solve for λ.

How? Using dot product

View $Ax = \lambda x$ as an overdetermined system of linear equations in λ.

minimize $\|Ax - \lambda x\|^2$ as a function of λ

$Ax = b$ by least squares for x

$x\lambda = Ax$ by least squares for λ

$Ax = b$ by least squares for x

$A^T A x = A^T b$

$x = (A^T A)^{-1} A^T b$

$L\phi(x) = -\lambda\sigma(x)\phi(x)$

Wrong way. Choose an i such that $x_i \neq 0$. Then

$[Ax]_i = \lambda x_i$

and $\quad \lambda = \dfrac{[Ax]_i}{x_i}$

Wrong way for functions

choose an x such that $\phi(x) \neq 0$. and then divide

$L\phi(x) = -\lambda\sigma(x)\phi(x)$

$\lambda = \dfrac{-L\phi(x)}{\sigma(x)\phi(x)}$

Same thing in red

$x\lambda = Ax$ by least squares for λ

$\boxed{x^T x}\ \lambda = \boxed{x^T A x}$

$\lambda = (x^T x)^{-1} x^T A x$ since $x \in \mathbb{R}^n$ then $x^T x = x \cdot x$

$= (x \cdot x)^{-1}\ x \cdot Ax$

$\lambda = \dfrac{x \cdot Ax}{x \cdot x}$ Rayleigh quotient:

Solve by least squares

$L\varphi(x) = -\lambda \sigma(x)\varphi(x)$

$\boxed{(\varphi, L\varphi)} = \boxed{(\varphi, -\lambda\sigma\varphi)} = -\lambda(\varphi, \sigma\varphi)$

$\lambda = -\dfrac{(\varphi, L\varphi)}{(\varphi, \sigma\varphi)} = \dfrac{-(\varphi, L\varphi)}{(\varphi,\varphi)_\sigma}$

note $(\varphi,\varphi)_\sigma > 0$ by last time ..

Since

$L\varphi = \frac{d}{dx}\left(p(x)\,\varphi'(x)\right) + q(x)\,\varphi(x)$

Then

$(\varphi, L\varphi) = \int_a^b \varphi(x)\,L\varphi(x)\,dx = \int_a^b \varphi(x)\left(\frac{d}{dx}\left(p(x)\,\varphi'(x)\right) + q(x)\,\varphi(x)\right)dx$

$(\varphi,\varphi)_\sigma = \int_a^b \varphi(x)\,\sigma(x)\,\varphi(x)\,dx = \boxed{\int_a^b \sigma(x)\,\varphi(x)^2\,dx}$

From last time

$$(\varphi,\varphi)_\sigma = \int_a^b \sigma(x)|\varphi(x)|^2\,dx \geqslant \int_{c-\delta}^{c+\delta} \frac{|f(c)|}{2}\,dx > 0$$

Now simplify this...

$$(\varphi, L\varphi) = \int_a^b \varphi(x)\,L\varphi(x)\,dx = \int_a^b \varphi(x)\left(\frac{d}{dx}\left(p(x)\,\varphi'(x)\right) + q(x)\,\varphi(x)\right)dx$$

$$= \underbrace{\int_a^b \varphi(x)\,\frac{d}{dx}\left(p(x)\,\varphi'(x)\right)dx}_{\text{integrate by parts}} + \underbrace{\boxed{\int_a^b \varphi(x)\,q(x)\,\varphi(x)\,dx}}_{\text{okay}}$$

$$\int_a^b \varphi(x)\,\frac{d}{dx}\left(p(x)\,\varphi'(x)\right)dx = p(x)\,\varphi(x)\varphi'(x)\Big|_a^b - \int_a^b p(x)(\varphi'(x))^2\,dx$$

$u = \varphi(x)$ $\qquad du = \varphi'(x)\,dx$

$dv = \frac{d}{dx}\left(p(x)\,\varphi'(x)\right)dx$ $\qquad v = p(x)\,\varphi'(x)$

$$\lambda = -\frac{(\varphi, L\varphi)}{(\varphi,\sigma\varphi)} = \frac{-\left(p(x)\,\varphi(x)\varphi'(x)\Big|_a^b - \int_a^b p(x)(\varphi'(x))^2\,dx + \int_a^b \varphi(x)\,q(x)\,\varphi(x)\,dx\right)}{\int_a^b \sigma(x)\,\varphi(x)^2\,dx}$$

$$\boxed{\lambda = \frac{-p(x)\,\varphi(x)\varphi'(x)\Big|_a^b + \int_a^b p(x)(\varphi'(x))^2\,dx - \int_a^b q(x)(\varphi(x))^2\,dx}{\int_a^b \sigma(x)\,\varphi(x)^2\,dx}}$$

$$\lambda = \frac{-p\phi \, d\phi/dx\big|_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2] \, dx}{\int_a^b \phi^2 \sigma \, dx},$$

Since $p > 0$ then $\int_a^b p(x)\left(\phi'(x)\right)^2 dx > 0$

Suppose that

a. There is a smallest eigenvalue, usually denoted λ_1.

was already known...

$$\lambda = \frac{-p(x)\phi(x)\phi'(x)\Big|_a^b + \int_a^b p(x)(\phi'(x))^2 \, dx - \int_a^b q(x)(\phi(x))^2 \, dx}{\int_a^b \sigma(x)\phi(x)^2 \, dx}$$

Minimization principle. The Rayleigh quotient cannot be used to determine explicitly the eigenvalue (since ϕ is unknown). Nonetheless, it can be quite useful in estimating the eigenvalues. This is because of the following theorem: **The minimum value of the Rayleigh quotient for all continuous functions satisfying the boundary conditions** (but not necessarily the differential equation) **is the lowest eigenvalue:**

$$\boxed{\lambda_1 = \min \frac{-pu \, du/dx\big|_a^b + \displaystyle\int_a^b \left[p\left(du/dx\right)^2 - qu^2\right] dx}{\displaystyle\int_a^b u^2 \sigma \, dx}, \qquad (5.6.5)}$$

where λ_1 represents the smallest eigenvalue. The minimization includes all continuous functions that satisfy the boundary conditions. The minimum is obtained only for $u = \phi_1(x)$, the lowest eigenfunction. For example, the lowest eigenvalue is important in heat flow problems (see Section 5.4).

Often interested in knowing that $(u, Lu) \geq 0$ for all u.
This is the same as the eigenvalues of L being positive...

3. Corresponding to each eigenvalue λ_n, there is an eigenfunction, denoted $\phi_n(x)$ (which is unique to within an arbitrary multiplicative constant).

Uniqueness of the eigenfunctions... depends on boundary conditions and wasn't true for some of the problems we already worked... In partular, the ones where both sine and cosine were eigenfunctions.

$$L\phi = \frac{d}{dx}\left(p(x)\,\phi'(x)\right) + q(x)\,\phi(x)$$

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$$\beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) = 0,$$

the PDE $\quad L\phi = -\lambda\sigma\phi$

Suppose there were two eigenfunctions ϕ_1 and ϕ_2 with the same eigenvalue

$$L\phi_1 = -\lambda\sigma\phi_1 \quad \text{and} \quad L\phi_2 = -\lambda\sigma\phi_2$$

Claim there is a constant c such that $\phi_1 = c\,\phi_2$.