$$\boxed{\underbrace{\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right)}_{L} + q(x)\phi + \lambda\sigma(x)\phi = 0,} \qquad a < x < b,$$

$$\boxed{\begin{aligned} \beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) &= 0 \\ \beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) &= 0, \end{aligned}}$$

and $p > 0$ and $\sigma > 0$

1. All the eigenvalues λ are real.
2. There exist an infinite number of eigenvalues:
$$\lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} < \ldots.$$
   a. There is a smallest eigenvalue, usually denoted λ_1.
   b. There is not a largest eigenvalue and $\lambda_n \to \infty$ as $n \to \infty$.
3. Corresponding to each eigenvalue λ_n, there is an eigenfunction, denoted $\phi_n(x)$ (which is unique to within an arbitrary multiplicative constant). $\phi_n(x)$ has exactly $n - 1$ zeros for $a < x < b$.
4. The eigenfunctions $\phi_n(x)$ form a "complete" set, meaning that any piecewise smooth function $f(x)$ can be represented by a generalized Fourier series of the eigenfunctions:
$$f(x) \sim \sum_{n=1}^{\infty} a_n\phi_n(x).$$
   Furthermore, this infinite series converges to $[f(x+) + f(x-)]/2$ for $a < x < b$ (if the coefficients a_n are properly chosen).
5. Eigenfunctions belonging to different eigenvalues are orthogonal relative to the weight function $\sigma(x)$. In other words,
$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx = 0 \quad \text{if } \lambda_n \neq \lambda_m.$$
6. Any eigenvalue can be related to its eigenfunction by the **Rayleigh quotient**:
$$\lambda = \frac{-p\phi\, d\phi/dx\big|_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2]\,dx}{\int_a^b \phi^2\sigma\,dx},$$
   where the boundary conditions may somewhat simplify this expression.

3. Corresponding to each eigenvalue λ_n, there is an eigenfunction, denoted $\phi_n(x)$ (which is unique to within an arbitrary multiplicative constant).

Suppose there were two eigenfunctions ϕ_1 and ϕ_2 with the same eigenvalue

$$L\phi_1 = -\lambda\sigma\phi_1 \quad \text{and} \quad L\phi_2 = -\lambda\sigma\phi_2$$

Claim there is a constant c such that $\phi_1 = c\phi_2$.

We know $L = L^\dagger$, that is that L is self adjoint

$$\boxed{(\phi_1, L\phi_2)} = (L^\dagger\phi_1, \phi_2) = (L\phi_1, \phi_2)$$

$$(\phi_1, -\lambda\sigma\phi_2) = (-\lambda\sigma\phi_1, \phi_2)$$ (obvious doesn't say anything)

Also

$$\int_a^b \boxed{\phi_1(x)\big(L\phi_2(x)\big)}dx = \int_a^b \big(L\phi_1(x)\big)\phi_2(x)\,dx$$

look a this on its own

Start over...

3. Corresponding to each eigenvalue λ_n, there is an eigenfunction, denoted $\phi_n(x)$ (which is unique to within an arbitrary multiplicative constant).

Suppose there were two eigenfunctions ϕ_1 and ϕ_2 with the same eigenvalue

$$L\phi_1 = -\lambda\sigma\phi_1 \qquad \text{and} \qquad L\phi_2 = -\lambda\sigma\phi_2$$

Claim there is a constant c such that $\boxed{\phi_1 = c\phi_2}$.

$$\phi_2 L \phi_1 = \boxed{-\lambda\sigma\phi_2\phi_1} \qquad \phi_1 L \phi_2 = \boxed{-\lambda\sigma\phi_1\phi_2}$$

same

So I get $\phi_2 L \phi_1 = \phi_1 L \phi_2$ ⇐ holds pointwise ... since the eigenvalue was the same ..

looks like this ↓ except without the integral.

$$\boxed{(\phi_1, L\phi_2)} = (L^{\dagger}\phi_1, \phi_2) = (L\phi_1, \phi_2)$$

Now

$$\phi_2 L \phi_1 = \phi_1 L \phi_2$$

$$L = \frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi$$

$$\phi_2(x)\left[\frac{d}{dx}\left(p(x)\,\phi_1'(x)\right) + q(x)\,\phi_1(x)\right] = \phi_1(x)\left[\frac{d}{dx}\left(p(x)\,\phi_2'(x)\right) + q(x)\,\phi_2(x)\right]$$

$$\phi_2(x)\frac{d}{dx}\left(p(x)\,\phi_1'(x)\right) + \cancel{q(x)\,\phi_1(x)\,\phi_2(x)} = \phi_1(x)\frac{d}{dx}\left(p(x)\,\phi_2'(x)\right) + \cancel{q(x)\phi_1(x)\,\phi_2(x)}$$

Thus $\phi_2(x)\frac{d}{dx}\left(p(x)\,\phi_1'(x)\right) = \phi_1(x)\,\frac{d}{dx}\left(p(x)\,\phi_2'(x)\right)$

or $$\phi_2(x)\frac{d}{dx}\left(p(x)\,\phi_1'(x)\right) - \phi_1(x)\frac{d}{dx}\left(p(x)\,\phi_2'(x)\right) = 0$$

$$\phi_2(x)\frac{d}{dx}\left(p(x)\,\phi_1'(x)\right) = \frac{d}{dx}\left(\phi_2(x)\,p(x)\,\phi_1'(x)\right) - \phi_2'(x)\,\cancel{p(x)}\,\phi_1'(x)$$

$$\phi_1(x)\frac{d}{dx}\left(p(x)\,\phi_2'(x)\right) = \frac{d}{dx}\left(\phi_1(x)\,p(x)\,\phi_2'(x)\right) - \phi_1'(x)\,\cancel{p(x)}\,\phi_2'(x)$$

Therefore

$$\frac{d}{dx}\left(\phi_2(x)\,p(x)\,\phi_1'(x)\right) - \frac{d}{dx}\left(\phi_1(x)\,p(x)\,\phi_2'(x)\right) = 0$$

$$\frac{d}{dx}\Big(\underbrace{p(x)\left(\phi_1'(x)\,\phi_2(x) - \phi_1(x)\,\phi_2'(x)\right)}_{\text{what's inside is constant}}\Big) = 0$$

Thus

$$p(x)\left(\phi_1'(x)\,\phi_2(x) - \phi_1(x)\,\phi_2'(x)\right) = \text{const.}$$

for all $x \in [a,b]$.

In particular for x on the boundary.

$$\boxed{\begin{aligned}\beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) &= 0\\ \beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) &= 0,\end{aligned}}$$

Use the boundary conditions to find the const.

$$\phi_1'(a) = \frac{\beta_1}{\beta_2}\phi_1(a)$$

$$\phi_2'(a) = \frac{\beta_1}{\beta_2}\phi_2(a)$$

$$p(a)\left(\phi_1'(a)\,\phi_2(a) - \phi_1(a)\,\phi_2'(a)\right) = p(a)\left[\frac{\beta_1}{\beta_2}\cancel{\phi_1(a)\,\phi_2(a)} - \cancel{\phi_1(a)\frac{\beta_1}{\beta_2}\phi_2(a)}\right] = 0$$

So the const. is 0,

Thus;

$$p(x)\left(\varphi_1'(x)\varphi_2(x) - \varphi_1(a)\varphi_2'(x)\right) = 0$$

since $p > 0$

$$\boxed{\varphi_1'(x)\varphi_2(x) - \varphi_1(a)\varphi_2'(x)} = 0$$

Consider

$$\frac{d}{dx}\left(\frac{\varphi_1}{\varphi_2}\right) = \frac{\varphi_1'\varphi_2 - \varphi_1\varphi_2'}{\varphi_2^2} = \frac{0}{\varphi_2^2} = 0$$

assuming $(\varphi_2(x))^2 \neq 0$.

Thus $\frac{\varphi_1}{\varphi_2}$ is constant, so $\frac{\varphi_1}{\varphi_2} = c$

or equivalently $\boxed{\varphi_1 = c\varphi_2}$

Recall the minimization principle:

$$\boxed{\lambda_1 = \min \frac{-pu\, du/dx\big|_a^b + \int_a^b \left[p\,(du/dx)^2 - qu^2\right]\, dx}{\int_a^b u^2\sigma\, dx},}$$

This came from the Rayleigh quotient.

$$\lambda_i = -\frac{(\varphi_i, L\varphi_i)}{(\varphi_i, \sigma\varphi_i)}$$

Thus

$$\lambda_1 = \min \left\{ \frac{-(\varphi, L\varphi)}{(\varphi, \sigma\varphi)} : \text{for all } \varphi \text{ that satisfy the boundary conditions} \right\}$$

Example:

ODE $\quad \dfrac{d^2\varphi}{dx^2} + \lambda\varphi = 0 \qquad L\varphi = \dfrac{d^2}{dx^2}\varphi$

B.C. $\quad \varphi(0) = 0 \qquad \varphi(1) = 0$

$$\boxed{\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0,} \qquad a < x < b,$$

$p(x) = 1$, $q(x) = 0$ and $\sigma(x) = 1$

$a = 0$ and $b = 1$

Know there is an basis of orthogonal eigenfunctions

① — ⑥

We've solved this before $\varphi'' = -\lambda\varphi$

General solution $\varphi(x) = a\cos\sqrt{\lambda}x + b\sin\sqrt{\lambda}x$

$\varphi(0) = a = 0$ so $a = 0$

$\varphi(1) = b\sin\sqrt{\lambda} = 0$ since $b \neq 0$

Then $\sin\sqrt{\lambda} = 0$ so $\sqrt{\lambda} = n\pi$ for $n = 1, 2, \ldots$

Eigen functions are $\varphi_n(x) = \sin n\pi x$

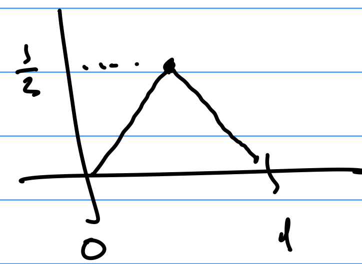
eigen values are $\lambda_n = n^2\pi^2$

What does this say?

$$\lambda_1 = \min \left\{ \frac{-(\varphi, L\varphi)}{(\varphi, \varphi)} : \text{for all } \varphi \text{ that satisfy the boundary conditions} \right\}$$

Let φ be the function

$$\varphi(x) = \begin{cases} x & \text{for } x < \frac{1}{2} \\ 1-x & \text{for } x \geq 1/2 \end{cases}$$



$$\pi^2 = \lambda_1 \leq \frac{-(\varphi, L\varphi)}{(\varphi, \varphi)}$$

$$= \frac{\cancel{-p\varphi\, d\varphi/dx|_a^b} + \int_0^1 \left[1\,(d\varphi/dx)^2 - \cancel{q\varphi^2}\right] dx}{\int_0^1 u^2 1\, dx}$$

$$= \frac{\int_0^1 (\varphi'(x))^2\, dx}{\int_0^1 (\varphi(x))^2\, dx} \quad \text{upper bound on } \lambda_1$$