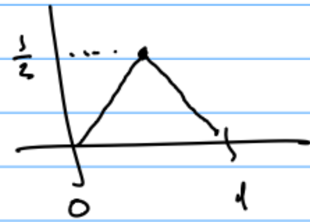
Let φ be the function

$$\varphi(x) = \begin{cases} x & \text{for } x < \frac{1}{2} \\ 1-x & \text{for } x \geq 1/2 \end{cases}$$



$$\pi^2 = \lambda_1 \leq \frac{-(\varphi, L\varphi)}{(\varphi, \varphi)}$$

$$= \frac{-\varphi\, d\varphi/dx\Big|_0^1 + \int_0^1 \left[(d\varphi/dx)^2 - 0\varphi^2 \right] dx}{\int_0^1 u^2 \varphi\, dx}$$

$$\lambda_1 \leq \frac{-\varphi(x)\varphi'(x)\Big|_0^1 + \int_0^1 \varphi'(x)^2\, dx}{\int_0^1 \varphi(x)^2\, dx}$$

where φ satisfies the boundary conditions

ODE $\quad \dfrac{d^2\varphi}{dx^2} + \lambda\varphi = 0 \qquad\qquad L\varphi = \dfrac{d^2}{dx^2}\varphi$

B.C. $\quad \varphi(0) = 0 \qquad \varphi(1) = 0$

Let

$$\varphi(x) = \begin{cases} x & \text{for } x < \frac{1}{2} \\ 1-x & \text{for } x \geq 1/2 \end{cases}$$

Then $\varphi(0) = 0$ and $\varphi(1) = 1 - 1 = 0$

Plug the function in to

$$\frac{-\varphi(x)\varphi'(x)\Big|_0^1 + \int_0^1 \varphi'(x)^2\,dx}{\int_0^1 \varphi(x)^2\,dx}$$

Since

$\varphi(x) = \begin{cases} x & \text{for } x < \frac{1}{2} \\ 1-x & \text{for } x \geq 1/2 \end{cases}$ then $\varphi'(x) = \begin{cases} 1 & \text{for } x < 1/2 \\ -1 & \text{for } x > 1/2 \end{cases}$

then

$$\varphi(x)\varphi'(x)\Big|_0^1 = \varphi(0)\varphi'(0) - \varphi(1)\varphi'(1) = 0\cdot 1 - 0\cdot(-1) = 0$$

$$\int_0^1 \varphi'(x)^2\,dx = \int_0^{1/2} (1)^2\,dx + \int_{1/2}^1 (-1)^2\,dx = 1$$

$$\int_0^1 \varphi(x)^2\,dx = \int_0^{1/2} x^2\,dx + \int_{1/2}^1 (1-x)^2\,dx = \frac{1}{3}x^3\Big|_0^{1/2} - \frac{1}{3}(1-x)^3\Big|_{1/2}^1$$

$$= \frac{1}{3\cdot 2^3} + \frac{1}{3\cdot 2^3} = \frac{1}{24} + \frac{1}{24} = \frac{1}{12}$$

Therefore

$$\pi^2 \leq \frac{-\varphi(x)\varphi'(x)\Big|_0^1 + \int_0^1 \varphi'(x)^2\,dx}{\int_0^1 \varphi(x)^2\,dx} = \frac{0+1}{1/12} = 12.$$

Another bound by using a different φ that satisfies the boundary conditions

$$\varphi(x) = x - x^2 = x(1-x) \qquad \varphi'(x) = 1-2x$$

$$\varphi(0) = 0 \quad \text{and} \quad \varphi(1) = 1$$

Now compute

$$\frac{\overbrace{-\varphi(x)\varphi'(x)\Big|_0^1}^{0} + \int_0^1 \varphi'(x)^2\,dx}{\int_0^1 \varphi(x)^2\,dx}$$

$$\int_0^1 \varphi'(x)^2\,dx = \int_0^1 (1-2x)^2\,dx = \frac{-1}{2}\,\frac{1}{3}(1-2x)^3\Big|_0^1$$

$$= -\frac{1}{6}\left[(-1)^3 - (1)^3\right] = \frac{2}{6} = \frac{1}{3}.$$

$$\int_0^1 (x-x^2)^2\,dx = \int_0^1 (x^2 - 2x^3 + x^4)\,dx = \frac{1}{3}x^3 - \frac{2}{4}x^4 + \frac{1}{5}x^5\Big|_0^1$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{1}{5} = \frac{-1}{6} + \frac{1}{5} = \frac{1}{30}.$$

Therefore

$$\pi^2 \le \frac{\int_0^1 \varphi'(x)^2\,dx}{\int_0^1 \varphi(x)^2\,dx} = \frac{1/3}{1/30} = 10$$

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julia> pi^2
9.869604401089358
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Note if $\varphi(x) = \sin \pi x$ then one gets equality...

Recall

$$\boxed{\underbrace{\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right)} + q(x)\phi + \lambda\sigma(x)\phi = 0,} \qquad a < x < b,$$

$$L\phi(x) = \frac{d}{dx}\left(p(x)\,\phi'(x)\right) + q(x)\,\phi(x)$$

Then $L\phi = -\lambda\sigma\phi$ is the PDE.

$$(\phi, L\phi) = -\lambda(\phi, \sigma\phi)$$

Solved for λ

$$\lambda = \frac{-(\phi, L\phi)}{(\phi, \sigma\phi)} = \frac{-(\phi, L\phi)}{(\phi,\phi)_\sigma}$$

Idea: If $u(x)$ satisfies the boundary conditions but not the PDE then

$$\lambda_1 = \min\left\{\frac{-(u, Lu)}{(u,u)_\sigma} \;;\; \boxed{u \text{ satisfies the boundary conditions}}\right\}$$

Use #4 to justify this...

4. The eigenfunctions $\phi_n(x)$ form a "complete" set, meaning that any piecewise smooth function $f(x)$ can be represented by a generalized Fourier series of the eigenfunctions:

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x).$$

Furthermore, this infinite series converges to $[f(x+) + f(x-)]/2$ for $a < x < b$ (if the coefficients a_n are properly chosen).

Let $u = \sum_{n=1}^{\infty} a_n \phi_n(x)$ and plug it in...

$$(u, Lu) = \left(\sum_{n=1}^{\infty} a_n \phi_n(x) , L \sum_{n=1}^{\infty} a_n \phi_n(x) \right)$$

$$= \left(\sum_{n=1}^{\infty} a_n \phi_n(x) , \sum_{n=1}^{\infty} a_n \, L \phi_n(x) \right)$$

these are solutions to the PDE so

$L\phi_n = -\lambda_n \sigma \phi_n$

$$= \left(\sum_{n=1}^{\infty} a_n \phi_n(x) , \sum_{m=1}^{\infty} a_m \left(-\lambda_m \sigma(x) \phi_m(x)\right) \right)$$

$$= -\int_a^b \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n a_m \lambda_m \sigma(x) \phi_n(x) \phi_m(x) \, dx$$

$$= -\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n a_m \lambda_m \boxed{\int_a^b \sigma(x) \phi_n(x) \phi_m(x) \, dx}$$

$$= -\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n a_m \lambda_m (\phi_n, \phi_m)_\sigma .$$

the only terms that survive are terms on the diagonal

by

5. Eigenfunctions belonging to different eigenvalues are orthogonal relative to the weight function $\sigma(x)$. In other words,

$$(\phi_n, \phi_m)_\sigma = \int_a^b \phi_n(x)\phi_m(x)\sigma(x) \, dx = 0 \quad \text{if } \lambda_n \neq \lambda_m.$$

$$(u, Lu) = -\sum_{n=1}^{\infty} a_n a_n \lambda_n (\phi_n, \phi_n)_\sigma$$

$$(u, Lu) = -\sum_{n=1}^{\infty} a_n^2 \lambda_n (\phi_n, \phi_n)_\sigma$$

Also

$$(u, u)_\sigma = \left(\sum_{n=1}^{\infty} a_n \phi_n, \sum_{n=1}^{\infty} a_n \phi_n \right)_\sigma$$

$$= \int_a^b \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n \phi_n(x)\, \sigma(x) a_m \phi_m(x)\, dx$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n a_m \int_a^b \sigma(x) \phi_n(x) \phi_m(x)\, dx$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n a_m (\phi_n, \phi_m)_\sigma = \sum_{n=1}^{\infty} a_n^2 (\phi_n, \phi_n)_\sigma$$

Thus

quantities being averaged

weights

$$\frac{-(u, Lu)}{(u, u)_\sigma} = \frac{\sum_{n=1}^{\infty} \lambda_n \boxed{a_n^2 (\phi_n, \phi_n)_\sigma}}{\sum_{n=1}^{\infty} \boxed{a_n^2 (\phi_n, \phi_n)_\sigma}}$$

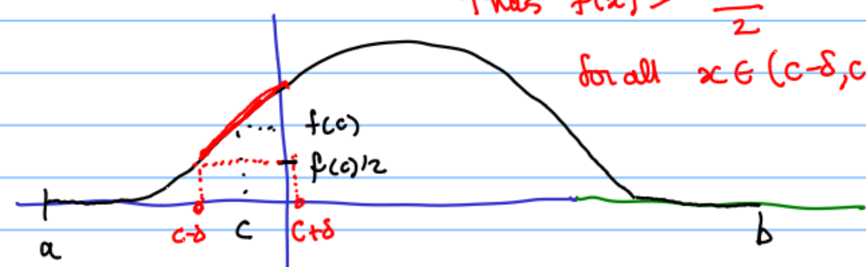
Interpret the above as a weighted average of the λ_n's

Why are $a_n^2 (\varphi_n, \varphi_n)_\sigma$ like weights? They need to be non-negative (actually positive).

$$a_n^2 (\varphi_n, \varphi_n)_\sigma = \underbrace{a_n^2}_{\geq 0} \underbrace{\int_a^b \sigma(x) (\varphi_n(x))^2 \, dx}_{>0} \geq 0$$

recall

Thus $f(x) > \frac{f(c)}{2}$ for all $x \in (c-\delta, c+\delta)$



Conclusion $\displaystyle\int_a^b \sigma(x) |\varphi(x)|^2 \, dx \geq \int_{c-\delta}^{c+\delta} \frac{|f(c)|}{2} \, dx > 0$

Note that some a_n's are positive since we assume $u \neq 0$. Since Average is greater than the minimum. Then

$$\lambda_1 = \min \lambda_n\text{'s} \leq \frac{\sum_{n=1}^{\infty} \lambda_n a_n^2 (\varphi_n, \varphi_n)_\sigma}{\sum_{n=1}^{\infty} a_n^2 (\varphi_n, \varphi_n)_\sigma}$$

HW Due on Friday