What's next?

Quiz 2 on April 19

HW 5 due today.

HW6 is extra and the last one... it will be used to replace any lower HW score if you turn it in.

PDE: $\boxed{\rho \dfrac{\partial^2 u}{\partial t^2} = T_0 \dfrac{\partial^2 u}{\partial x^2}}$ BC: $\boxed{\begin{array}{c} u(0,t) = 0 \\ u(L,t) = 0 \end{array}}$ IC: $\boxed{\begin{array}{c} u(x,0) = f(x) \\ \dfrac{\partial u}{\partial t}(x,0) = g(x). \end{array}}$

$\rho = \rho(x)$ so the density is no longer constant.

Use separation of variables an superposition to solve the PDE.

$$u(x,t) = \varphi(x)\, h(t).$$

Thus

$$\rho(x)\, \varphi(x)\, h''(t) = T_0\, \varphi''(x)\, h(t)$$

$$\frac{h''(t)}{h(t)} = \frac{T_0}{\rho(x)} \frac{\varphi''(x)}{\varphi(x)} = -\lambda \quad \leftarrow \text{doesn't depend on } x \text{ or } t.$$

only a func of t only a funct of x

We get 2 ODEs.

$$h''(t) = -\lambda h(t) \quad \text{and} \quad T_0 \varphi''(x) = -\lambda \rho(x) \varphi(x)$$

$$\varphi(0) = 0 \quad \text{and} \quad \varphi(L) = 0$$

Compare with

PDE

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0, \qquad a < x < b,$$

eigenvalues

$L = \frac{d}{dx} p(x) \frac{d}{dx} + q(x)$

$L\phi = \frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi$

BC

$$\beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) = 0$$

$$\beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) = 0,$$

and $p > 0$ and $\sigma > 0$

$T_0\varphi''(x) = -\lambda\rho(x)\varphi(x)$

$\varphi'(0) = 0$ and $\varphi(L) = 0$

$a = 0 \quad b = L$

$\beta_1 = 1, \ \beta_2 = 0$

$\beta_3 = 1 \quad \beta_4 = 0$

So this ODE obtained from seperation of variables is a Sturm-Liouville problem... Therefore all of the following holds:

1. All the eigenvalues λ are real.
2. There exist an infinite number of eigenvalues:
$$\lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} < \ldots .$$
   a. There is a smallest eigenvalue, usually denoted λ_1.
   b. There is not a largest eigenvalue and $\lambda_n \to \infty$ as $n \to \infty$.
3. Corresponding to each eigenvalue λ_n, there is an eigenfunction, denoted $\phi_n(x)$ (which is unique to within an arbitrary multiplicative constant). $\phi_n(x)$ has exactly $n-1$ zeros for $a < x < b$.
4. The eigenfunctions $\phi_n(x)$ form a "complete" set, meaning that any piecewise smooth function $f(x)$ can be represented by a generalized Fourier series of the eigenfunctions:
$$f(x) \sim \sum_{n=1}^{\infty} a_n\phi_n(x).$$
Furthermore, this infinite series converges to $[f(x+) + f(x-)]/2$ for $a < x < b$ (if the coefficients a_n are properly chosen).
5. Eigenfunctions belonging to different eigenvalues are orthogonal relative to the weight function $\sigma(x)$. In other words,
$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx = 0 \quad \text{if } \lambda_n \neq \lambda_m.$$
6. Any eigenvalue can be related to its eigenfunction by the **Rayleigh quotient**:
$$\lambda = \frac{-p\phi\, d\phi/dx|_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2]\,dx}{\int_a^b \phi^2\sigma\,dx},$$
where the boundary conditions may somewhat simplify this expression.

$p(x) = T_0 > 0$
$\sigma(x) = \rho(x) > 0$
$q(x) = 0$

Let's look at the Rayleigh quotient

$\varphi(0)=0$ and $\varphi(L)=0$

$$\lambda = \frac{-p\phi \, d\phi/dx\big|_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2] \, dx}{\int_a^b \phi^2 \sigma \, dx},$$

$$\lambda = \frac{-T_0 \,\varphi(x)\, \varphi'(x)\Big|_0^L + \int_0^L T_0^2 (\varphi'(x))^2 \, dx}{\int_0^L (\varphi(x))^2 \rho(x) dx} = \frac{\int_0^L T_0^2 (\varphi'(x))^2 \, dx}{\int_0^L (\varphi(x))^2 \rho(x) dx}$$

Note that the integrals are always positive, so $\lambda > 0$.

Since $\lambda > 0$ we can now solve

$$h''(t) = -\lambda h(t)$$

in general as

$$h(t) = c_1 \cos\sqrt{\lambda}\, t + c_2 \sin\sqrt{\lambda}\, t$$ ... where λ is an eigenvalue of the Sturm-Liouville problem for φ.

---

4. The eigenfunctions $\phi_n(x)$ form a "complete" set, meaning that any piecewise smooth function $f(x)$ can be represented by a generalized Fourier series of the eigenfunctions:

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x).$$

Furthermore, this infinite series converges to $[f(x+) + f(x-)]/2$ for $a < x < b$ (if the coefficients a_n are properly chosen).

Let λ_n, φ_n be the eigenvalue-eigenfunction pairs corresponding to the Sturm-Liouville problem. and by superposition write

$$u(x,t) = \sum_{n=1}^{\infty} a_n \left(\cos\sqrt{\lambda_n}\, t\right) \varphi_n(x) + b_n \left(\sin\sqrt{\lambda_n}\, t\right) \varphi_n(x)$$

Now solve for the a_n and b_n in terms of

$$\boxed{\text{IC:} \quad \begin{aligned} u(x,0) &= f(x) \\ \frac{\partial u}{\partial t}(x,0) &= g(x). \end{aligned}}$$

$$u(x,0) = \sum_{n=1}^{\infty} a_n \phi_n(x) = f(x)$$

. Eigenfunctions belonging to different eigenvalues are orthogonal relative to the weight function $\sigma(x)$. In other words,

$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx = 0 \quad \text{if } \lambda_n \neq \lambda_m.$$

Need to add the weight $\sigma(x) = \rho(x)$ to use orthogonality to solve for the coefficients. Therefore,

$$\sum_{n=1}^{\infty} a_n \phi_n(x)\phi_m(x)\rho(x) = f(x)\phi_m(x)\rho(x)$$

Now integrate

$$\int_0^L \sum_{n=1}^{\infty} a_n \phi_n(x)\phi_m(x)\rho(x)\,dx = \int_0^L f(x)\phi_m(x)\rho(x)\,dx$$

$$\sum_{n=1}^{\infty} a_n \underbrace{\int_0^L \phi_n(x)\phi_m(x)\rho(x)\,dx}_{=0 \text{ when } m \neq m} = \int_0^L f(x)\phi_m(x)\rho(x)\,dx$$

$$\boxed{\frac{\int_0^L f(x)\phi_n(x)\rho(x)\,dx}{\int_0^L \phi_n^2 \rho\,dx}} = a_n$$

$$a_m \int_0^L (\phi_m(x))^2 \rho(x)\,dx = \int_0^L f(x)\phi_m(x)\rho(x)\,dx$$

Thus

$$(*) \qquad a_m = \int_0^L f(x)\phi_m(x)\rho(x)\,dx \Big/ \int_0^L (\phi_m(x))^2 \rho(x)\,dx$$

Now solve for the other initial condition and the b_n's.

$$u(x,t) = \sum_{n=1}^{\infty} a_n \left(\cos\sqrt{\lambda_n}\, t\right)\varphi_n(x) + b_n\left(\sin\sqrt{\lambda_n}\, t\right)\varphi_n(x)$$

$$\frac{\partial u}{\partial t}(x,0) = g(x).$$

$$u_t(x,t) = \sum_{n=1}^{\infty} -a_n\sqrt{\lambda_n}\left(\sin\sqrt{\lambda_n}\, t\right)\varphi_n(x) + b_n\sqrt{\lambda_n}\left(\cos\sqrt{\lambda_n}\, t\right)\varphi_n(x)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} b_n\sqrt{\lambda_n}\,\varphi_n(x) = g(x)$$

Multiply by the weight and use orthogonality

$$\int_0^L \sum_{n=1}^{\infty} b_n\sqrt{\lambda_n}\,\varphi_n(x)\,\varphi_m(x)\,\rho(x)\,dx = \int_0^L g(x)\,\varphi_m(x)\,\rho(x)\,dx$$

$$b_m\sqrt{\lambda_m}\int_0^L (\varphi_m(x))^2\rho(x)\,dx = \int_0^L g(x)\,\varphi_m(x)\,\rho(x)\,dx$$

Therefore

$$b_n\sqrt{\lambda_n} = \frac{\int_0^L g(x)\phi_n(x)\rho(x)\,dx}{\int_0^L \phi_n^2\rho\,dx},$$

$$(**)\quad b_m = \int_0^L g(x)\,\varphi_m(x)\,\rho(x)\,dx \Big/ \sqrt{\lambda_m}\int_0^L (\varphi_m(x))^2\rho(x)\,dx$$

Solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \left(\cos\sqrt{\lambda_n}\, t\right)\varphi_n(x) + b_n\left(\sin\sqrt{\lambda_n}\, t\right)\varphi_n(x)$$

where the a_n and b_m are determined by $(*)$ and $(**)$ and the λ_n, φ_n come from the Sturm-Liouville problem.