Finding the sign error

$$-\int_a^{x_s(t)} \frac{\partial}{\partial t}\rho(x,t)\,dx = \int_a^{x_s(t)} c(\rho)\frac{\partial}{\partial x}\rho(x,t)\,dx = \int_a^{x_s(t)} \frac{\partial q}{\partial x}\,dx = \boxed{q(x_s(t)^-)} - q(a)$$

$$-\int_{x_s(t)}^{b} \frac{\partial}{\partial t}\rho(x,t)\,dx = \int_{x_s(t)}^{b} c(\rho)\frac{\partial}{\partial x}\rho(x,t)\,dx = \int_{x_s(t)}^{b} \frac{\partial q}{\partial x}\,dx = q(b) \boxed{- q(x_s(t)^+)}$$

$$\frac{dx_s(t)}{dt} = \frac{\cancel{q(a,t)} - \cancel{q(b,t)} \cancel{+} \boxed{q(x_s(t)^+)} - \cancel{q(a)} + \cancel{q(b)} + \boxed{q(x_s(t)^-)}}{\rho(x_s(t)^-,t) - \rho(x_s(t)^+,t)}$$

Therefore

$$\frac{dx_s(t)}{dt} = \frac{q(x_s(t)^-,t) - q(x_s(t)^+,t)}{\rho(x_s(t)^-,t) - \rho(x_s(t)^+,t)}$$

<u>Example:</u> 1. $\frac{\partial \rho}{\partial t} + 2\rho\frac{\partial \rho}{\partial x} = 0 \qquad \rho(x,0) = \begin{cases} 4 & \text{for } x \le 0 \\ 3 & \text{for } x > 0 \end{cases}$

Solve this nonlinear (quasi linear) PDE using characteristics and the assumption that ρ is conserved.

Parameterize x, t. In this case use t as the parameter.

~~Let~~ $x = x(t)$

$$\frac{d}{dt}\rho(x(t),t) = \frac{\partial \rho}{\partial t} + x'(t)\frac{\partial \rho}{\partial x}$$

Gives two ODEs

$$x'(t) = 2\rho \qquad \text{and} \qquad \frac{d}{dt}\rho(x(t),t) = 0$$

Solve $\frac{d}{dt}\rho(x(t),t) = 0$ first

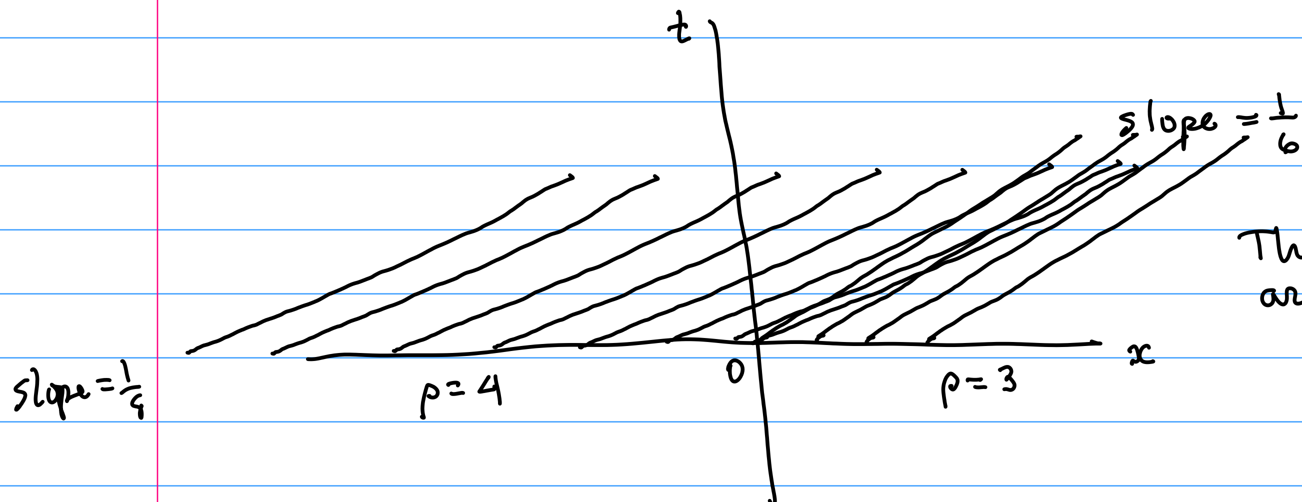
$$\rho(x(t),t) = \text{const} = \rho(x_0, 0) = \begin{cases} 4 & \text{for } x_0 \leq 0 \\ 3 & \text{for } x_0 > 0 \end{cases}$$

Solve

$$x'(t) = 2\rho(x_0, 0)$$

const of integration $x_0 = x(0)$.

$$\boxed{x(t) = 2t\rho(x_0,0) + x_0}$$



These characteristics are crossing ...

overlapping characteristics and need to use the jump condition to understand how to preserve ρ as the shock evolves.

What is q?

$$\frac{dx_s(t)}{dt} = \frac{q(x_s(t)^-,t) - q(x_s(t)^+,t)}{\rho(x_s(t)^-,t) - \rho(x_s(t)^+,t)}$$

recall

$$\frac{\partial q}{\partial x} = c(\rho)\frac{\partial \rho}{\partial x} = 2\rho\frac{\partial \rho}{\partial x}$$

$$q = \int \frac{dq}{\cancel{dx}}\cancel{dx} = \int 2\rho \frac{d\rho}{\cancel{dx}}\cancel{dx} = \int 2\rho\, d\rho = \rho^2$$

why?

$$\frac{dq}{dx} = 2\rho\frac{d\rho}{dx} = \frac{d\rho^2}{dx} \quad \text{ODE in } x.$$

so $q = \rho^2 + \text{const}$.

const $= 0$

$$\frac{dx_s(t)}{dt} = \frac{\rho^2(x_s(t)^-,t) - \rho^2(x_s(t)^+,t)}{\rho(x_s(t)^-,t) - \rho(x_s(t)^+,t)}$$

$$= \frac{4^2 - 3^2}{4-3} = 7$$

$$\rho(x,0) = \begin{cases} 4 & \text{for } x \le 0 \\ 3 & \text{for } x > 0 \end{cases}$$

$$\frac{dx_s}{dt} = 7$$

$$x_s(t) = 7t + c = 7t$$

$$\rho(x,t) = \begin{cases} 4 & \text{for } x \le x_s(t) \\ 3 & \text{for } x > x_s(t) \end{cases}$$

Therefore the solution is

$$\boxed{\rho(x,t) = \begin{cases} 4 & \text{for } x \le 7t \\ 3 & \text{for } x > 7t \end{cases}}$$

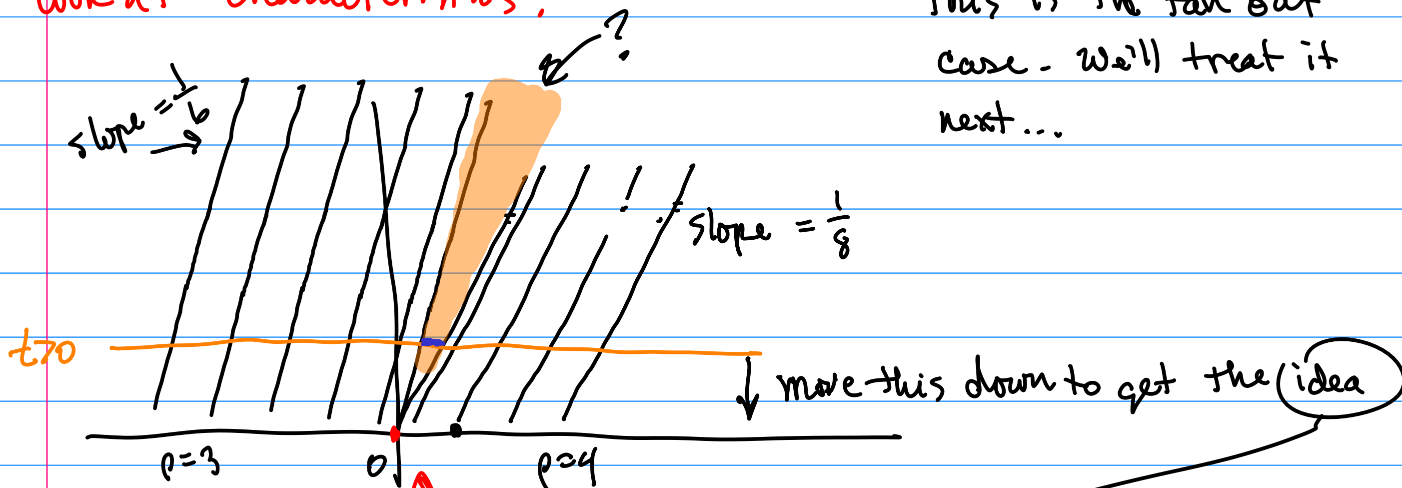
✓

What about the case

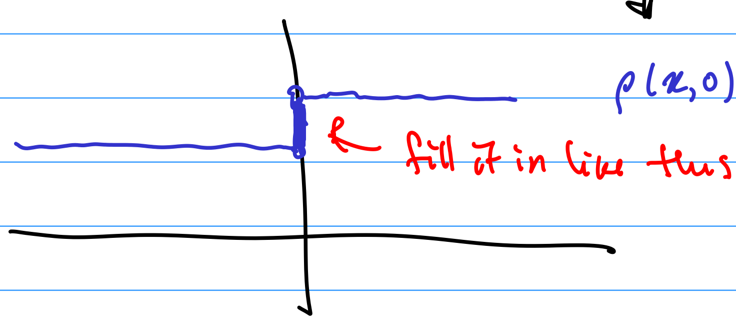
Example: 1. $\frac{\partial \rho}{\partial t} + 2\rho \frac{\partial \rho}{\partial x} = 0$ $\rho(x,0) = \begin{cases} 3 & \text{for } x \leq 0 \\ 4 & \text{for } x > 0 \end{cases}$

look at characteristics.

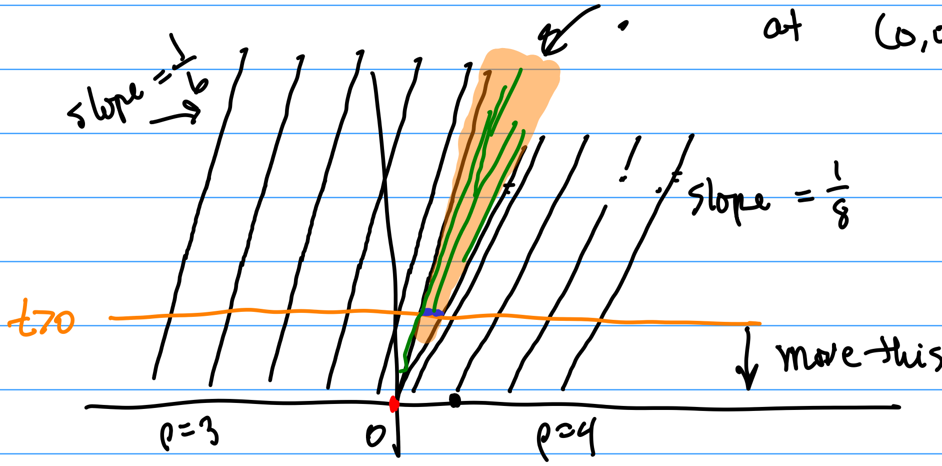
This is the fan out case. We'll treat it next...



idea. Imagine that within this initial jump at $t=0$ all values of ρ between 3 and 4 are concentrated at $x=0$.



fill the gap by characteristics of all possible slopes between 3 and 4 starting at (0,0)



$$x(t) = 2\rho t + 0 \quad \text{for} \quad \rho \in [3,4].$$

$$\rho = \frac{x}{2t}$$

Solution $\rho(x,t) = \begin{cases} 3 & \text{for} \quad x \le 6t \\ \frac{x}{2t} & \text{for} \quad 6t < x \le 8t \\ 4 & \text{for} \quad x > 8t \end{cases}$