We are solving the ODE:

Therefore we have the ODE.

$$\varphi''(x) = -\lambda\varphi(x)$$

$$\varphi(0)=0 \quad \text{and} \quad \varphi(L)=0$$

Case $\lambda=0$. No non-zero solutions

Case $\lambda<0$ No non-zero solutions.

Case $\lambda>0$.

$$\varphi'' = -\lambda\varphi \quad \xrightarrow{\text{substitute}} \quad \left\{\begin{array}{l} \varphi = e^{rx} \\ \varphi' = re^{rx} \\ \varphi'' = r^2e^{rx} \end{array}\right.$$

$$r^2 \cancel{e^{rx}} = -\lambda \cancel{e^{rx}}$$

$r^2 = -\lambda$ so $r = \pm\sqrt{-\lambda} = \pm i\sqrt{\lambda}$

since λ is positive...

$$\left\{\begin{array}{l} e^{i\sqrt{\lambda}x} = \cos\sqrt{\lambda}x + i\sin\sqrt{\lambda}x \\ e^{-i\sqrt{\lambda}x} = \cos\sqrt{\lambda}x - i\sin\sqrt{\lambda}x \end{array}\right\}$$

instead we just remember the general solution to the ODE is

$$\varphi(x) = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x$$

Boundary conditions

$$\varphi(0)=0 \quad \text{and} \quad \varphi(L)=0$$

$$\varphi(0) = c_1 \cos 0 + c_2 \sin 0 = c_1 = 0$$

$$\varphi(L) = \underbrace{c_2 \sin\sqrt{\lambda}\, L = 0}$$

this means $\sqrt{\lambda}\, L = n\pi$ for some $n \in \mathbb{Z}$.

Thus $\sqrt{\lambda} = \frac{n\pi}{L}$

Therefore

$$\varphi(x) = c_2 \sin \frac{n\pi}{L} x$$

satisfies the ODE + Boundary conditions.

Then $u(x,t) = c_2\left(\sin \frac{n\pi}{L} x\right) G(t)$

constant... how to solve for that and what about $G(t)$?

We have the other ODE

$G' = -\lambda k G$ and initial condition

$u(x,0) = f(x)$ for all $x \in [0, L]$.

Let's solve the ODE first...

$\sqrt{\lambda} = \frac{n\pi}{L}$ so $\lambda = \frac{n^2\pi^2}{L^2}$

Thus

$$G' = -\frac{n^2\pi^2}{L^2} k G$$

General solution is

$$G(t) = c_3 e^{-\frac{n^2\pi^2}{L^2}kt}$$

Then

$$u(x,t) = c_2\left(\sin\frac{n\pi}{L}x\right) c_3 e^{-\frac{n^2\pi^2}{L^2}kt}$$

$$u_n(x,t) = B_n\left(\sin\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}kt} \quad \text{for } n \in \mathbb{Z}$$

This is a bunch of function which satisfy the PDE + Boundary conditions, but not the initial condition.

If the initial distribution of heat is very special then one of the solutions u_n and a suitable choice of B_n might satisfy that initial cond.

Since the PDE is linear and the Boundary conditions homogeneous, then sums of solutions also satisfy the PDE + Boundary.

$$u(x,t) = \sum_{n\in\mathbb{Z}} B_n\left(\sin\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}kt}$$

Try to solve for B_n such that $u(x,0) = f(x)$

$$\sum_{n\in\mathbb{Z}} B_n\left(\sin\frac{n\pi}{L}x\right) = f(x)$$

Note that when $-n$ is substituted for n then

$$u_{-n}(x,t) = B_{-n}\left(\sin\frac{-n\pi}{L}x\right) e^{-\frac{(-n)^2\pi^2}{L^2}kt}$$

$$= -B_{-n}\left(\sin\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}kt}$$

That the same as setting $B_n = -B_{-n}$ in u_n so I don't need the negative terms in the sum.

Thus

$$u(x,t) = \sum_{n\in\mathbb{N}} B_n\left(\sin\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}kt}$$

Try to solve for B_n such that $u(x,0) = f(x)$

$$\sum_{n\in\mathbb{N}} B_n\left(\sin\frac{n\pi}{L}x\right) = f(x)$$

To solve for the B_n we need the theory of Fourier series...

The sine functions have an orthogonality property that makes solving this equations for B_n easy

$$\int_0^L \underbrace{\sin\frac{n\pi}{L}x \sin\frac{m\pi}{L}x \, dx}_{} =$$

integrate by parts twice and then solve for the integral...

or use some trigonometry and integrate just once →

Angle addition

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$

$$\frac{d}{da}\sin(a+b) = \frac{d}{da}(\sin a \cos b + \cos a \sin b)$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

subtract

$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$

$$\cos(a+b) - \cos(a-b) = -2\sin a \sin b$$

$$\sin a \sin b = \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right)$$

$$\int_0^L \sin\frac{n\pi}{L}x \,\sin\frac{m\pi}{L}x \, dx$$

$$= \frac{1}{2}\int_0^L \left[\cos\frac{(n-m)\pi}{L}x - \cos\frac{(n+m)\pi}{L}x\right] dx$$

Case $n = m$

$$\frac{1}{2}\int_0^L \left(1 - \cos\frac{2\pi n}{L}x\right) dx$$

$$= \frac{L}{2} - \sin\frac{2\pi n x}{L}\Big|_0^L = \frac{L}{2}$$