$$A_0 + \sum_{n=1} \left(A_n \cos\frac{n\pi}{L}x + B_n \sin\frac{n\pi}{L}x\right) = f(x).$$

lots of orthogonality...

addition

$$\left.\begin{aligned} \sin(a+b) &= \sin a\cos b + \cos a \sin b \\ \sin(a-b) &= \sin a \cos b - \cos a \sin b \end{aligned}\right\}$$

add them

$\sin a \cos b = \frac{1}{2}(\sin(a+b) + \sin(a-b))$

$$\frac{d}{da}\sin(a+b) = \frac{d}{da}(\sin a \cos b + \cos a \sin b)$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

trust $\cos(a-b) = \cos a \cos b + \sin a \sin b$

$$\int_{-L}^{L} \left(\cos\frac{n\pi}{L}x\right)\left(\sin\frac{m\pi}{L}x\right)dx = \int_{-L}^{L} \frac{1}{2}\left(\sin\frac{(n+m)\pi}{L}x + \sin\frac{(n-m)\pi}{L}x\right)dx$$

$n=m$ then $\sin\frac{(n-m)\pi}{L}x = 0$

Case $n=m$

$$\int_{-L}^{L} \frac{1}{2}\sin\frac{2n\pi}{L}x = \frac{1}{2}\frac{L}{2n\pi}\cos\frac{2n\pi}{L}x\Big|_{-L}^{L} = 0$$

Case $n \neq m$

$$\int_{-L}^{L} \frac{1}{2}\left(\sin\frac{(n+m)\pi}{L}x + \sin\frac{(n-m)\pi}{L}\right)$$

$n+m \neq 0$ because $n,m \geq 0$

$$= \frac{1}{2}\frac{L}{(n+m)\pi}\cos\frac{(n+m)\pi}{L}x\Big|_{-L}^{L} + \frac{1}{2}\frac{L}{(n-m)\pi}\cos\frac{(n-m)\pi}{L}x\Big|_{-L}^{L} = 0$$

$$A_0 + \sum_{n=1}^{\infty} \left(A_n \cos\frac{n\pi}{L}x + B_n \sin\frac{n\pi}{L}x\right) = f(x).$$

To find A_0 just integrate

$$\int_{-L}^{L}\left(A_0+\sum_{n=1}^{\infty}\left(A_n \cancel{\cos\frac{n\pi}{L}x}+B_n \cancel{\sin\frac{n\pi}{L}x}\right)\right)dx=\int_{-L}^{L}f(x)\,dx$$

$$2L A_0=\int_{-L}^{L}f(x)\,dx$$

$$A_0=\frac{1}{2L}\int_{-L}^{L}f(x)\,dx$$

To find A_m with $m \geq 1$ mult. by $\cos\frac{m\pi}{L}x$.

$$\left(\cos\frac{m\pi}{L}x\right)\left(A_0+\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi}{L}x+B_n \cancel{\sin\frac{n\pi}{L}x}\right)\right)=f(x)\cos\frac{m\pi}{L}x$$

$$\int_{-L}^{L}\left(\sum_{n=1}^{\infty}A_n\cos\frac{m\pi}{L}x\cos\frac{n\pi}{L}x\right)dx=\int_{-L}^{L}\left(\cos\frac{m\pi}{L}x\right)f(x)\,dx$$

$$2L\frac{1}{2}A_m=\int_{-L}^{L}\left(\cos\frac{m\pi}{L}x\right)f(x)\,dx$$

$$A_m=\frac{1}{L}\int_{-L}^{L}\left(\cos\frac{m\pi}{L}x\right)f(x)\,dx$$

Similarly

$$B_m=\frac{1}{L}\int_{-L}^{L}\left(\sin\frac{m\pi}{L}x\right)f(x)\,dx$$

Summarize.

PDE $\quad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ for $x \in [-L, L]$ and $t \geq 0$

B.C. $\quad u(-L) = u(L)$

$\frac{\partial u}{\partial x}(-L) = \frac{\partial u}{\partial x}(L)$ for $t \geq 0$

I.C. $\quad u(x,0) = f(x)$ for $x \in [-L, L]$.

The solution is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi}{L}x + B_n \sin\frac{n\pi}{L}x\right)$$

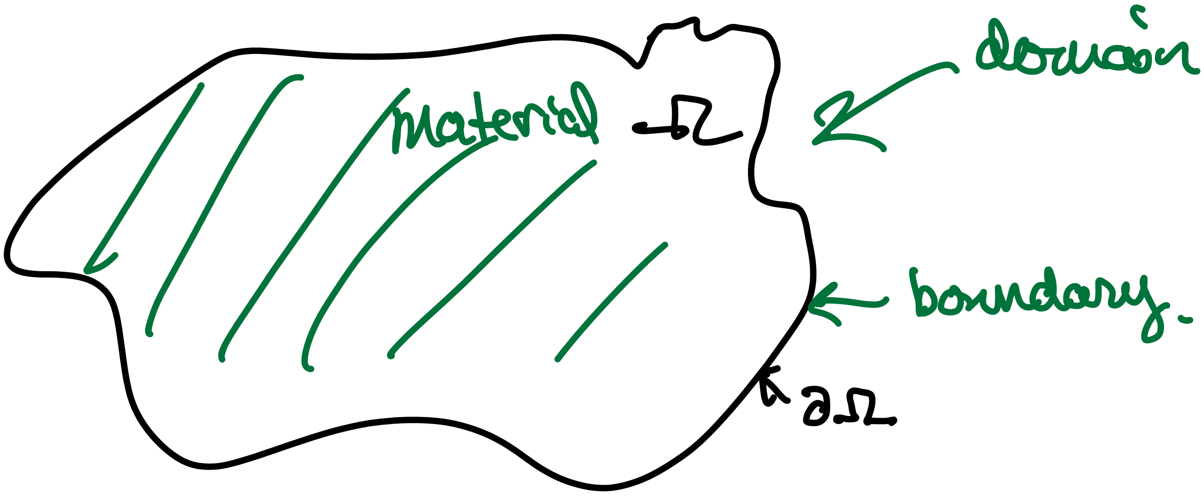
where

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \quad A_n = \frac{1}{L}\int_{-L}^{L}\left(\cos\frac{n\pi}{L}x\right)f(x)\,dx$$

$$B_n = \frac{1}{L}\int_{-L}^{L}\left(\sin\frac{n\pi}{L}x\right)f(x)\,dx$$

Heat equation in 2D.

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$ for $(x,y) \in \Omega$ and $t \geq 0$

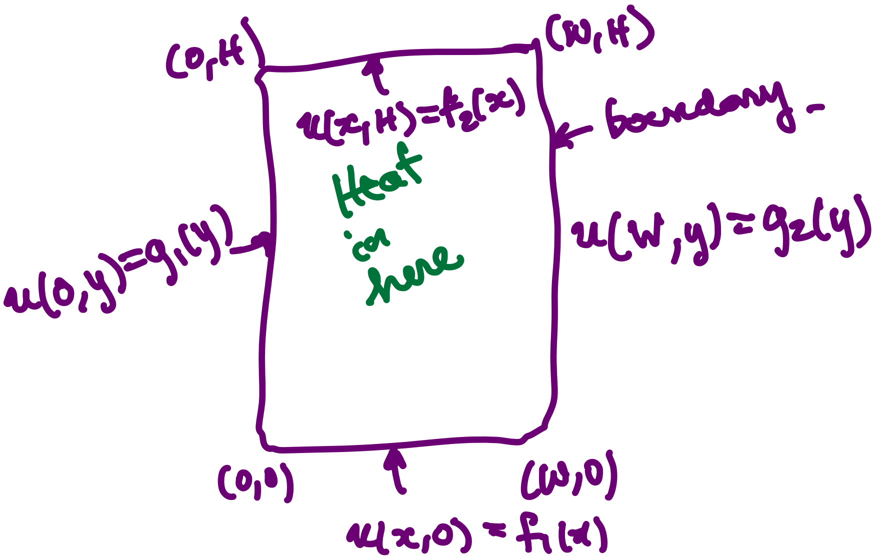


What about solutions as $t \to \infty$? that is equilibrium solution when $\frac{\partial u}{\partial t} = 0$.
Laplaces equation

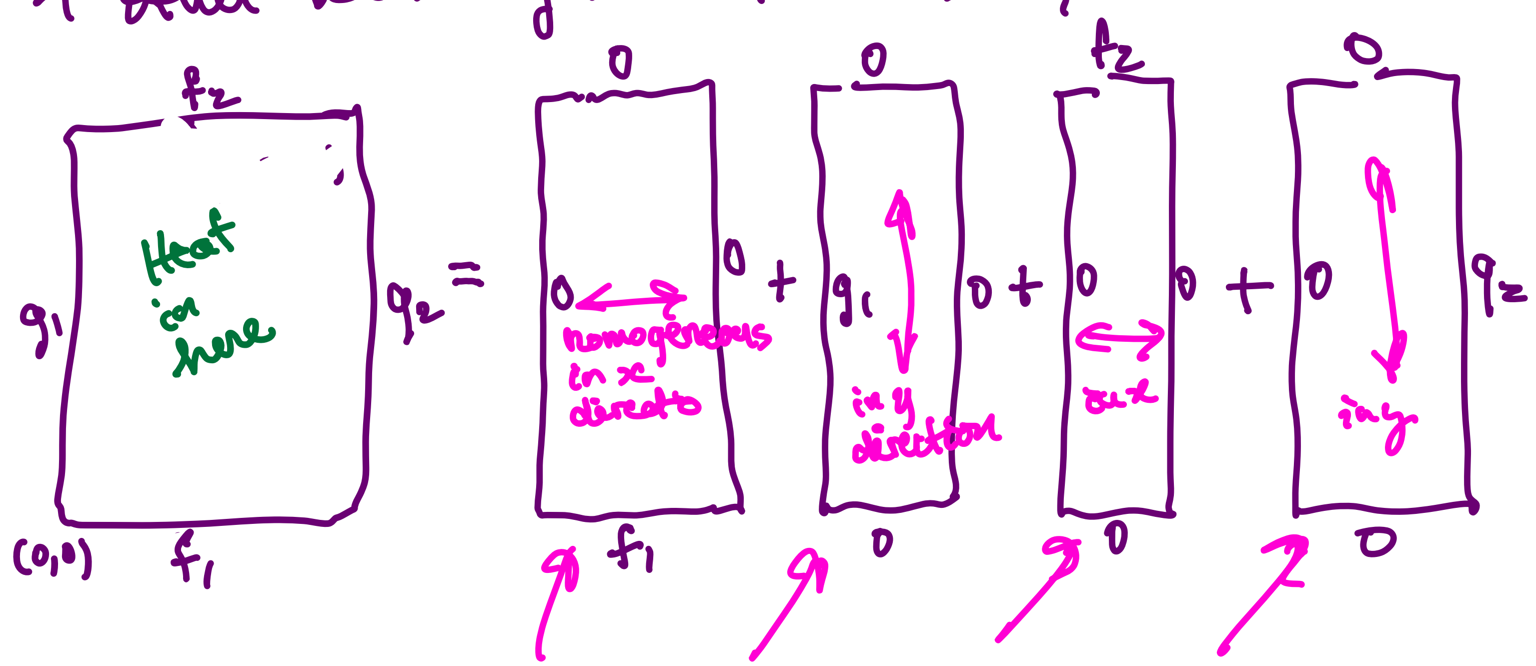
$$0 = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad (x,y) \in \Omega.$$

Still a PDE. Linear ... try separation of variables to create 2 ODEs. one in x and one in y.

Consider $\Omega = [0,W] \times [0,H]$



To manage the fact that none of the boundaries are homogeneous, we write this problem as a sum of 4 other boundary value problems



Solve each of these using separation of variables and then add them together to solve the problem with inhomogeneous boundary

Quiz on Friday.

RCISES 1.4

1.4.1. Determine the equilibrium temperature distribution for a one-dimensional rod with constant thermal properties with the following sources and boundary conditions:

* (a)	$Q = 0$,	$u(0) = 0$,	$u(L) = T$
(b)	$Q = 0$,	$u(0) = T$,	$u(L) = 0$
(c)	$Q = 0$,	$\frac{\partial u}{\partial x}(0) = 0$,	$u(L) = T$
* (d)	$Q = 0$,	$u(0) = T$,	$\frac{\partial u}{\partial x}(L) = \alpha$
(e)	$\frac{Q}{K_0} = 1$,	$u(0) = T_1$,	$u(L) = T_2$
* (f)	$\frac{Q}{K_0} = x^2$,	$u(0) = T$,	$\frac{\partial u}{\partial x}(L) = 0$
(g)	$Q = 0$,	$u(0) = T$,	$\frac{\partial u}{\partial x}(L) + u(L) = 0$
* (h)	$Q = 0$,	$\frac{\partial u}{\partial x}(0) - [u(0) - T] = 0$,	$\frac{\partial u}{\partial x}(L) = \alpha$

Heat equation

From last time

$$c(\cancel{x})\,\rho(\cancel{x})\,\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(K_0(\cancel{x})\,\frac{\partial u}{\partial x}\right) + Q(x,t)$$

General heat

Then

$$\cancel{c\rho\frac{\partial u}{\partial t}} = K_0\frac{\partial^2 u}{\partial x^2} + Q$$

$\frac{\partial u}{\partial t}=0$

(e) $\frac{Q}{K_0} = 1$ $\quad u(0) = T_1$ and $u(L) = T_2$

Then, $\frac{d^2u}{dx^2} = -\frac{Q}{K_0} = -1$

general solution to the ODE $u'' = -1$.

$$u' = \int -1\,dx = -1x + c_1$$

$$u = \int(-1x + c_1)\,dx = -\frac{x^2}{2} + c_1x + c_2$$

Boundary condition:

$$u(0) = -\frac{x^2}{2} + c_1x + c_2\Big|_{x=0} = c_2 = T_1$$

$$u(L) = -\frac{L^2}{2} + c_1L + T_1 = T_2$$

Thus $$c_1 L = T_2 + \frac{L^2}{2} - T_1$$

$$c_1 = \frac{T_2 - T_1 + \frac{L^2}{2}}{L} = \frac{T_2 - T_1}{L} + \frac{L}{2}$$

Answer

$$u(x) = -\frac{x^2}{2} + \left(\frac{T_2 - T_1}{L} + \frac{L}{2}\right)x + T_1$$