$$r = \frac{-\beta \pm \sqrt{\beta^2 - 4\rho_0 \lambda_n T_0}}{2\rho_0},$$

$\lambda_n = \left(\frac{n\pi}{L}\right)^2$ $\quad n = 1, 2, \ldots$

$$h(t) = c_1 e^{\frac{-\beta + \sqrt{\beta^2 - 4\rho_0\lambda_n T_0}}{2\rho_0} t} + c_2 e^{\frac{-\beta - \sqrt{\beta^2 - 4\rho_0\lambda_n T_0}}{2\rho_0} t}$$

assume $\beta^2 < 4\pi^2 \rho_0 T_0 / L^2$

You can assume that this frictional coefficient β is relatively small ($\beta^2 < 4\pi^2 \rho_0 T_0/L^2$).

$$\beta^2 - 4\rho_0\lambda_n T_0 = \beta^2 - 4\rho_0 \left(\frac{n\pi}{L}\right)^2 T_0 < 4\pi^2 \rho_0 T_0 / L^2 - 4\rho_0\left(\frac{n\pi}{L}\right)^2 T_0$$

$$= \frac{4\rho_0 T_0 \pi^2}{L^2}\left(1 - n^2\right) \lesssim 0$$

Therefore $\beta^2 - 4\rho_0\lambda_n T_0 < 0$ for every $n = 1, 2, \ldots$

Thus $\sqrt{\beta^2 - 4\rho_0\lambda_n T_0} = i\sqrt{\underbrace{4\rho_0\lambda_n T_0 - \beta^2}_{\text{positive}}}$

and

$$h(t) = c_1 e^{\frac{-\beta + i\sqrt{4\rho_0\lambda_n T_0 - \beta^2}}{2\rho_0} t} + c_2 e^{\frac{-\beta - i\sqrt{4\rho_0\lambda_n T_0 - \beta^2}}{2\rho_0} t}$$

recall $e^{i\theta} = \cos\theta + i\sin\theta$ (because the Taylor series are the same)

$$e^{\frac{-\beta + i\sqrt{4\rho_0\lambda_n T_0 - \beta^2}}{2\rho_0}t} = e^{\frac{-\beta}{2\beta}t} \; e^{i\frac{\sqrt{4\rho_0\lambda_n T_0-\beta^2}}{2\rho_0}t}$$

$$= e^{\frac{-\beta}{2\beta}t}\left(\cos\frac{\sqrt{4\rho_0\lambda_n T_0-\beta^2}}{2\rho_0}t + i\sin\frac{\sqrt{4\rho_0\lambda_n T_0-\beta^2}}{2\rho_0}t\right)$$

same

same

$$e^{\frac{-\beta - i\sqrt{4\rho_0\lambda_n T_0 - \beta^2}}{2\rho_0}t}$$

$$= e^{\frac{-\beta}{2\beta}t}\left(\cos -\frac{\sqrt{4\rho_0\lambda_n T_0-\beta^2}}{2\rho_0}t + i\sin -\frac{\sqrt{4\rho_0\lambda_n T_0-\beta^2}}{2\rho_0}t\right)$$

$$= e^{\frac{-\beta}{2\beta}t}\left(\cos\frac{\sqrt{4\rho_0\lambda_n T_0-\beta^2}}{2\rho_0}t - i\sin\frac{\sqrt{4\rho_0\lambda_n T_0-\beta^2}}{2\rho_0}t\right)$$

Therefore

These would have to be complex if $h(t)$ were to be real

$$h(t) = c_1 e^{\frac{-\beta + i\sqrt{4\rho_0\lambda_n T_0 - \beta^2}}{2\rho_0}t} + c_2 e^{\frac{-\beta - i\sqrt{4\rho_0\lambda_n T_0 - \beta^2}}{2\rho_0}t}$$

becomes

$$h_n(t) = e^{\frac{-\beta}{2\beta}t}\left(a_n\cos\frac{\sqrt{4\rho_0\lambda_n T_0-\beta^2}}{2\rho_0}t + b_n\sin\frac{\sqrt{4\rho_0\lambda_n T_0-\beta^2}}{2\rho_0}t\right)$$

real numbers because the displacement of the string is physical so real.

The superposition is

$$u(x,t) = \sum_{n=1}^{\infty} g_n(x)\, h_n(t)$$

$$= \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\, e^{\frac{-\beta}{2\beta}t}\left(a_n \cos\frac{\sqrt{4\rho_0\lambda_n T_0-\beta^2}}{2\rho_0}t + b_n \sin\frac{\sqrt{4\rho_0\lambda_n T_0-\beta^2}}{2\rho_0}t \right)$$

now use orthogonality to solve for the initial conditions...

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

$$u_t(x,t) = \frac{\partial}{\partial t}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\, e^{\frac{-\beta}{2\beta}t}\left(a_n \cos\frac{\sqrt{4\rho_0\lambda_n T_0-\beta^2}}{2\rho_0}t + b_n \sin\frac{\sqrt{4\rho_0\lambda_n T_0-\beta^2}}{2\rho_0}t \right)$$

$$= \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(\frac{-\beta}{2\rho_0}\right) e^{\frac{-\beta}{2\beta}t}\left(a_n \cos\frac{\sqrt{4\rho_0\lambda_n T_0-\beta^2}}{2\rho_0}t + b_n \sin\frac{\sqrt{4\rho_0\lambda_n T_0-\beta^2}}{2\rho_0}t \right)$$

$$+ \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\, e^{\frac{-\beta}{2\beta}t}\left(-a_n \frac{\sqrt{4\rho_0\lambda_n T_0-\beta^2}}{2\rho_0} \sin\frac{\sqrt{4\rho_0\lambda_n T_0-\beta^2}}{2\rho_0}t + b_n \frac{\sqrt{4\rho_0\lambda_n T_0-\beta^2}}{2\rho_0} \cos\frac{\sqrt{4\rho_0\lambda_n T_0-\beta^2}}{2\rho_0}t \right).$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\overbrace{\left[\left(\frac{-\beta}{2\rho_0}\right) a_n + b_n \frac{\sqrt{4\rho_0\lambda_n T_0-\beta^2}}{2\rho_0}\right]}^{\text{coefficients in sine series}} = g(x)$$

plug a_n's here and solve for b_n's.

$$u(x,0) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\, a_n = f(x)$$

solve for a_n here.

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$\left[\left(\frac{-\beta}{2\rho_0}\right) a_n + b_n \frac{\sqrt{4\rho_0\lambda_n T_0-\beta^2}}{2\rho_0}\right] = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

$$b_n \frac{\sqrt{4\rho_0 \lambda_n T_0 - \beta^2}}{2\rho_0} = \left(\frac{\beta}{2\rho_0}\right) a_n + \frac{2}{L}\int_0^L g(x) \sin\frac{n\pi x}{L}\, dx$$

$$b_n = \frac{2\rho_0}{\sqrt{4\rho_0 \lambda_n T_0 - \beta^2}} \left(\left(\frac{\beta}{2\rho_0}\right) a_n + \frac{2}{L}\int_0^L g(x) \sin\frac{n\pi x}{L}\, dx \right)$$

Final solution:

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\, e^{-\frac{\beta}{2\rho} t} \left(a_n \cos\frac{\sqrt{4\rho_0 \lambda_n T_0 - \beta^2}}{2\rho_0} t + b_n \sin\frac{\sqrt{4\rho_0 \lambda_n T_0 - \beta^2}}{2\rho_0} t \right)$$

where $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and

$$a_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$$

$$b_n = \frac{2\rho_0}{\sqrt{4\rho_0 \lambda_n T_0 - \beta^2}} \left(\left(\frac{\beta}{2\rho_0}\right) a_n + \frac{2}{L}\int_0^L g(x) \sin\frac{n\pi x}{L}\, dx \right)$$

Another example.

4.4.9. From (4.4.1), derive conservation of energy for a vibrating string,

$$\frac{dE}{dt} = c^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\bigg|_0^L, \tag{4.4.15}$$

where the total energy E is the sum of the kinetic energy, defined by $\int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx$, and the potential energy, defined by $\int_0^L \frac{c^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx$.

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},}$$

(4.4.1)

$$E = KE + PE = \int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx, + \int_0^L \frac{c^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx.$$

$$\frac{dE}{dt} = \frac{d}{dt}\left(\int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx, + \int_0^L \frac{c^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx\right)$$ put derivative inside

$$\frac{dE}{dt} = \int_0^L \frac{\partial}{\partial t}\left(\frac{1}{2}(u_t)^2 + \frac{c^2}{2}(u_x)^2\right) dx$$

$$= \int_0^L \left[\frac{1}{2}\frac{\partial}{\partial t}(u_t)^2 + \frac{c^2}{2}\frac{\partial}{\partial t}(u_x)^2\right] dx = \int_0^L \left[\frac{1}{2}\cdot 2 u_t u_{tt} + \frac{c^2}{2} 2 u_x u_{xt}\right] dx$$

$$u_{tt} = c^2 u_{xx}$$

$$= \int_0^L \left[u_t c^2 u_{xx} + c^2 u_x u_{xt}\right] dx = c^2 \int_0^L \left[u_t u_{xx} + u_x u_{xt}\right] dx$$

$$\frac{\partial}{\partial x}(u_t u_x) = u_{tx} u_x + u_t u_{xx}$$

$$= c^2 \int_0^L \frac{\partial}{\partial x}(u_t u_x)\, dx = c^2 u_t u_x \Big|_0^L$$