inner product

Dot product $x, y \in \mathbb{R}^n$ then $x \cdot y = \sum_{i=1}^{n} x_i y_i = x^T y$

outer product is xy^T but never mind

Let $A \in \mathbb{R}^{m \times n}$ a matrix means a linear function

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

defined $f(x) = Ax$ by matrix-vector multiplication. (actualy matrix-vector mult. is defined so f is a linear func).

If $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m \times n}$ then $Ax \in \mathbb{R}^m$

Take $y \in \mathbb{R}^m$ then y and Ax have length m.

$$Ax \cdot y = (Ax)^T y = x^T A^T y = x \cdot A^T y.$$

Review A^T is the matrix obtained by switching the columns of A with its rows. Note $A \in \mathbb{R}^{m \times n}$ implies $A^T \in \mathbb{R}^{n \times m}$.

Since $A^T \in \mathbb{R}^{n \times m}$ then A^T means a linear function

$$f^* : \mathbb{R}^m \to \mathbb{R}^n$$

defined $f^*(y) = A^T y$ by matrix-vector multiplication. Note that f^* maps in the other direction. This is the idea of duality...

If $m = n$ then A is square and we can ask the question whether $A^T = A$. If so, then

$$Ax \cdot y = x \cdot A^T y = x \cdot Ay \quad \text{for every } x \text{ and } y.$$

Suppose $Ax \cdot y = x \cdot Ay$ for every x and y. Does it then follow that $A = A^T$?

$$Ae_j = \left[a_1 \vdots a_2 \vdots \cdots \vdots a_n \right] e_j = a_j$$

$$Ae_j \cdot e_i = a_j \cdot e_i = \text{the } i\text{th component of } a_j$$

$$A = \left[a_1 \vdots a_2 \vdots \cdots \vdots a_n \right] = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & & A_{2n} \\ \vdots & & & \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

$$a_j = \begin{bmatrix} A_{1j} \\ A_{2j} \\ \vdots \\ A_{nj} \end{bmatrix} \qquad Ae_j \cdot e_i = a_j \cdot e_i = A_{ij}$$

I know $Ax \cdot y = x \cdot Ay$ for every x and y

and so $Ae_j \cdot e_i = e_j \cdot Ae_i$ for every i and j

$$A_{ij} = Ae_j \cdot e_i = Ae_i \cdot e_j = A_{ji}$$

Note if I know how $Ax \cdot y$ behaves for every vector x and y then we know what A is exactly.

<u>Spectral Theorem for Symmetric matrices</u>

If $A \in \mathbb{R}^{n \times n}$ and $A^T = A$ then there is an orthonormal basis of eigenvectors of A and the eigenvalues are real.

Thus, there are $v_i \in \mathbb{R}^n$ and $\lambda_i \in \mathbb{R}$ for $i = 1, \ldots, n$

such that

$$Av_i = \lambda_i v_i \quad \text{and} \quad v_i \cdot v_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}.$$

Correspondence to PDEs

Linear Algebra	PDE
vector	differentiable function with boundary conditions
matrix	differential operator
dot product $x \cdot y = \sum_{i=1}^{n} x_i y_i$	inner product $(f,g) = \int_0^L f(x)g(x)\,dx$
symmetric $A^T = A$, $Ax \cdot y = x \cdot Ay$ for all x, y	self-adjoint $L^{\dagger} = L$, $(Lf, g) = (f, Lg)$ for all f, g

Claim $Lf = f''$ is self adjoint for differentiable functions $f: [0,L] \to \mathbb{R}$ with $f(0)=0$ and $f(L)=0$.

$$(Lf, g) = \int_0^L f''(x)g(x)\,dx = uv\Big|_0^L - \int_0^L v\,du$$

$u = g(x)$ $\quad du = g'(x)\,dx$

$dv = f''(x)\,dx$ $\quad v = f'(x)$

$$= g(x)f'(x)\Big|_0^L - \int_0^L f'(x)g'(x)\,dx$$

$$= \underbrace{g(L)}_{=0} f'(L) - \underbrace{g(0)}_{=0} f'(0) - \int_0^L f'(x)\, g'(x)\, dx$$

$$= - \int_0^L f'(x) g'(x) dx$$

$u = g'(x)$ $\quad du = g''(x)\, dx$

$dv = f'(x)\, dx$ $\quad v = f(x)$

finish this next time...