Claim $Lf = f''$ is self adjoint for differentiable functions $f: [0,L] \to \mathbb{R}$ with $f(0)=0$ and $f(L)=0$.

$$(Lf, g) = \boxed{-\int_0^L f'(x) g'(x)\, dx}$$

are equal

$$(f, Lg) = \int_0^L f(x)\, Lg(x)\, dx = \int_0^L Lg(x)\, f(x)\, dx = (Lg, f)$$

now from before with the roles of f and g reversed

$$(Lg, f) = -\int_0^L g'(x) f'(x)\, dx = -\int_0^L f'(x) g'(x)\, dx$$

Therefore $(Lf, g) = (f, Lg)$ an so L is self adjoint, that is $L^\dagger = L$.

$$\boxed{\underbrace{\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi}_{L\phi(x)} + \lambda\sigma(x)\phi = 0,} \qquad a < x < b, \qquad \boxed{\begin{aligned} \beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) &= 0 \\ \beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) &= 0, \end{aligned}}$$

1. All the eigenvalues λ are real.
2. There exist an infinite number of eigenvalues:

$$\lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} < \ldots.$$

   a. There is a smallest eigenvalue, usually denoted λ_1.

   b. There is not a largest eigenvalue and $\lambda_n \to \infty$ as $n \to \infty$.
3. Corresponding to each eigenvalue λ_n, there is an eigenfunction, denoted $\phi_n(x)$ (which is unique to within an arbitrary multiplicative constant). $\phi_n(x)$ has exactly $n-1$ zeros for $a < x < b$.
4. The eigenfunctions $\phi_n(x)$ form a "complete" set, meaning that any piecewise smooth function $f(x)$ can be represented by a generalized Fourier series of the eigenfunctions:

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x).$$

   Furthermore, this infinite series converges to $[f(x+) + f(x-)]/2$ for $a < x < b$ (if the coefficients a_n are properly chosen).
5. Eigenfunctions belonging to different eigenvalues are orthogonal relative to the weight function $\sigma(x)$. In other words,

$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\, dx = 0 \quad \text{if } \lambda_n \neq \lambda_m.$$

6. Any eigenvalue can be related to its eigenfunction by the **Rayleigh quotient**:

$$\lambda = \frac{-p\phi\, d\phi/dx\big|_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2]\, dx}{\int_a^b \phi^2 \sigma\, dx},$$

   where the boundary conditions may somewhat simplify this expression.

$$\boxed{\underbrace{\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi}_{L\phi(x)} + \lambda\sigma(x)\phi = 0,} \qquad a < x < b,$$

weight is like a diagonal matrix

$$\underbrace{L\phi(x) = -\lambda\,\sigma(x)\,\phi(x)}_{\text{eigenfunction-eigenvalue equation for } L.}$$

ODE written using an operator

$$Ax = \lambda \Sigma x$$

diagonal matrix. (In 333 we have $\Sigma = I$).

Lagrange Identity... Is L self adjoint?

Does $(Lu, v) = (u, Lv)$ for all u, v satisfying the boundary cond.

$$\int_a^b Lu(x)\, v(x)\, dx = \int_a^b u(x)\, Lv(x)\, dx$$

$$\int_a^b \underbrace{\left(u(x)\, Lv(x) - v(x)\, Lu(x) \right)}_{\text{Simplify}} dx = ?$$

$$L\varphi(x) = \frac{d}{dx}\left(p(x) \frac{d\varphi(x)}{dx} \right) + q(x)\varphi(x)$$

$$u(x)\, Lv(x) - v(x)\, Lu(x)$$

$$= u(x) \left(\frac{d}{dx}\left(p(x) \frac{dv(x)}{dx} \right) + \cancel{q(x)\, v(x)} \right)$$

$$- v(x) \left(\frac{d}{dx}\left(p(x) \frac{du(x)}{dx} \right) + \cancel{q(x)\, u(x)} \right)$$

$$u(x)\, Lv(x) - v(x)\, Lu(x) = u(x) \frac{d}{dx}\left(p(x) \frac{dv(x)}{dx} \right) - v(x) \left(\frac{d}{dx}\left(p(x) \frac{du(x)}{dx} \right) \right)$$

... finish next time