$$\left(\frac{\partial u}{\partial x}\right)^2 \neq \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u^n}{\partial x} = n u^{n-1} \cdot \frac{\partial u}{\partial x}$$

$$\frac{\partial u^2}{\partial x} = 2 u^{2-1} \cdot \frac{\partial u}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)^2 \neq \frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial x}\right)^2$$

$$\boxed{\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)^2 = 2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}}$$

same thing...

$$\frac{\partial}{\partial x}\left(u_x^2\right) = 2 u_x u_{xx}$$

$$\underbrace{\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi}_{} + \lambda\sigma(x)\phi = 0,$$

$$\beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) = 0$$

$$\beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) = 0,$$

$$L\varphi(x) = \frac{d}{dx}\left(p(x)\frac{d\varphi(x)}{dx}\right) + q(x)\varphi(x)$$

$$L\varphi = (p\varphi')' + q\varphi$$

$$uLv - vLu = u\left[(pv')' + \cancel{qv}\right] - v\left[(pu')' + \cancel{qu}\right]$$

$$= u(pv')' - v(pu')'$$

Since

$$(upv')' = u'pv' + u(pv')'$$

$$(vpu')' = v'pu' + v(pu')'$$

Thus,

$$uLv - vLu = (upv')' - \cancel{u'pv'} - \left((vpu')' - \cancel{v'pu'}\right)$$

$$= (upv')' - (vpu')' = (upv' - vpu')'$$

$$uLv - vLu = \frac{d}{dx}\left(p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right)$$

Does $(Lu, v) = (u, Lv)$ for all u, v satisfying the boundary cond.

$$\int_a^b \underbrace{\left(u(x)\,Lv(x) - v(x)\,Lu(x)\right)}\,dx = \int_a^b \frac{d}{dx}\left(p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right)dx$$

$$\int_a^b \underbrace{\left(u(x)\,Lv(x) - v(x)\,Lu(x)\right)}\,dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b$$

functions evaluated at the boundary...

$$\beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) = 0$$

$$\beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) = 0,$$

Some of the constants are not zero..

Can't have $\beta_1 = \beta_2 = 0$ and can't have $\beta_3 = \beta_4 = 0$

$$p\left(u\frac{dv}{dx}-v\frac{du}{dx}\right)\Big|_a^b = p(b)u(b)v'(b)-p(b)v(b)u'(b) - \Big(p(a)u(a)v'(a)-p(a)v(a)u'(a)\Big)$$

Suppose $\beta_2 \neq 0$ then $\varphi'(a) = -\frac{\beta_1}{\beta_2}\varphi(a)$

$u'(a) = -\frac{\beta_1}{\beta_2}u(a)$ $\quad v'(a) = -\frac{\beta_1}{\beta_2}v(a)$

$$p(a)u(a)v'(a)-p(a)v(a)u'(a) = p(a)u(a)\cancel{\left(-\frac{\beta_1}{\beta_2}v(a)\right)} - p(a)v(a)\cancel{\left(-\frac{\beta_1}{\beta_2}u(a)\right)} = 0$$

Suppose $\beta_2 = 0$ then $\varphi(a)=0$, $u(a)=0$ and $v(a)=0$

$$p(a)u(a)v'(a)-p(a)v(a)u'(a) = 0-0=0$$

Similar argument shows that

$$p(b)u(b)v'(b)-p(b)v(b)u'(b)=0$$

when $\beta_4 \neq 0$ and also when $\beta_4 = 0$.

Therefore

$$\int_a^b \big(u(x)\,Lv(x) - v(x)\,Lu(x)\big)\,dx = 0-0=0$$

provided u and v satisfy the boundary conditions

So $(Lu,v) = (u,Lv)$

which means $L = L^\dagger$

or that L is self adjoint.

$$\beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) = 0$$
$$\beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) = 0,$$

$H = \{ \phi : [a,b] \to \mathbb{R}$ such that ϕ is differentiable and

$$\beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) = 0$$
$$\beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) = 0, \}$$

$(Lu,v) = (u,Lv)$ for all $u,v \in H$.

, and σ i.
l $\sigma > 0$ (analogy with this.

$Ax = \lambda \Sigma_\omega x$

diagonal matrix. with positive entries...

Spectral Theorem for Symmetric matrices

If $A \in \mathbb{R}^{n\times n}$ and $A^T = A$ then there is an orthonormal basis of eigenvectors of A and the eigenvalues are rea

Thus, there are $v_i \in \mathbb{R}^n$ and $\lambda_i \in \mathbb{R}$ for $i = 1, \ldots, n$

such that

$$Av_i = \lambda_i v_i \quad \text{and} \quad v_i \cdot v_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j. \end{cases}$$

no Σ here want to add Σ?

$$\Sigma = \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_n \end{bmatrix} \quad \text{with } \sigma_i > 0$$

$$\Sigma^{1/2} = \begin{bmatrix} \sqrt{\sigma_1} & & & 0 \\ & \sqrt{\sigma_2} & & \\ & & \ddots & \\ 0 & & & \sqrt{\sigma_n} \end{bmatrix}$$

$$Ax = \lambda \Sigma x$$

diagonal matrix.

$$Ax = \lambda \Sigma^{1/2} \underbrace{\Sigma^{1/2} x}_{y}$$

$$y = \Sigma^{1/2} x$$

$$x = \Sigma^{-1/2} y$$

$$A \Sigma^{-1/2} y = \lambda \Sigma^{1/2} y = \Sigma^{1/2} \lambda y$$

$$\underbrace{\Sigma^{-1/2} A \Sigma^{-1/2}}_{B} y = \lambda y$$

$By = \lambda y$ and

$$B^T = (\Sigma^{-1/2} A \Sigma^{-1/2})^T = (\Sigma^{-1/2})^T A^T (\Sigma^{-1/2})^T$$

$$= \Sigma^{-1/2} A \Sigma^{-1/2} = B$$

So B is symmetric and there is an orthonormal basis of eigenvectors y_i and eigenvalues λ_i to the matrix B.