$$Ax = \lambda \Sigma x = \lambda \Sigma^{1/2} \Sigma^{1/2} x$$

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diagonal matrix with positive entries...

$$\Sigma = \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_n \end{bmatrix} \qquad \Sigma^{-1/2} = \begin{bmatrix} \sigma_1^{1/2} & & & 0 \\ & \sigma_2^{1/2} & & \\ & & \ddots & \\ 0 & & & \sigma_n^{1/2} \end{bmatrix}$$

$$y = \Sigma^{1/2} x = \begin{bmatrix} \sigma_1^{1/2} & & & 0 \\ & \sigma_2^{1/2} & & \\ & & \ddots & \\ 0 & & & \sigma_n^{1/2} \end{bmatrix} x = \begin{bmatrix} \sigma_1^{1/2} x_1 \\ \sigma_2^{1/2} x_2 \\ \vdots \\ \sigma_n^{1/2} x_n \end{bmatrix}$$

$$x = \Sigma^{-1/2} y$$

λ is a scalar

$$A \Sigma^{-1/2} y = \lambda \Sigma^{1/2} y = \Sigma^{1/2} \lambda y$$

$$\Sigma^{-1/2} A \Sigma^{-1/2} y = \cancel{\Sigma^{-1/2}} \cancel{\Sigma^{1/2}} \lambda y$$

$$By = \lambda y \quad \text{where} \quad B = \Sigma^{-1/2} A \Sigma^{-1/2}$$

Now A symmetric implies B is symmetric

$$B^T = \left(\Sigma^{-1/2} A \Sigma^{-1/2} \right)^T =$$

transposes of the product in reverse order

same matrix so it doesn't matter if I reverse the order. And $A^T = A$ by hypothesis.

$$= \left(\Sigma^{-1/2} \right)^T A^T \left(\Sigma^{-1/2} \right)^T = \Sigma^{-1/2} A \Sigma^{-1/2} = B$$

So B is symmetric and there is an orthonormal basis of eigenvectors y_i and eigenvalues λ_i to the matrix B.

Thus there are $y_i \in \mathbb{R}^n$ and $\lambda_i \in \mathbb{R}$ such that

$$B y_i = \lambda_i y_i \quad \text{for all } i = 1, \dots, n$$

$$y_i \cdot y_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Rewrite this with matrix A and vectors x_i.

Define a weighted inner product

$$(u, v)_\Sigma = u \cdot \Sigma v = \Sigma^{1/2} u \cdot \Sigma^{1/2} v = \sum_{i=1}^{n} \sigma_i u_i v_i$$

Since $y_i = \Sigma^{1/2} x_i$ then

$$y_i \cdot y_j = \Sigma^{1/2} x_i \cdot \Sigma^{1/2} x_j = (x_i, x_j)_\Sigma$$

Therefore

$$(x_i, x_j)_\Sigma = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

By the definition $A x_i = \lambda_i \Sigma x_i$ for $i = 1, \dots, n$.

5. Eigenfunctions belonging to different eigenvalues are orthogonal relative to the weight function $\sigma(x)$. In other words,

$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\, dx = 0 \quad \text{if } \lambda_n \neq \lambda_m.$$

I know these are solutions to the ODE

$$\underbrace{\frac{d}{dx}\left(p(x)\frac{d\phi_n}{dx}\right) + q(x)\phi_n}_{L\phi_n} + \lambda_n\sigma(x)\phi_n = 0,$$

from before...

$$A x_i = \lambda_i \Sigma x_i$$

or $\quad L\phi_n = -\lambda_n \sigma(x)\phi_n$

$$L\phi_m = -\lambda_m \sigma(x)\phi_m$$

Let suppose $\lambda_n \neq \lambda_m$. Claim that

$$(\phi_n, \phi_m)_\sigma = \int_a^b \phi_n(x)\,\phi_m(x)\,\sigma(x)\,dx = 0$$

Recall Lagrange identity... $u = \phi_n$ and $v = \phi_m$

$$uLv - vLu = \frac{d}{dx}\left(p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right)$$

$$\phi_n L\phi_m - \phi_m L\phi_n = \frac{d}{dx}\left(p\left(\phi_n\frac{d\phi_m}{dx} - \phi_m\frac{d\phi_n}{dx}\right)\right)$$

$$(\phi_n)\underbrace{(-\lambda_m\sigma\phi_m)} - (\phi_m)\underbrace{(-\lambda_n\sigma\phi_n)}$$

$$\sigma(x)\,\phi_n(x)\,\phi_m(x)\,(\lambda_n - \lambda_m) = \frac{d}{dx}\left(p\left(\phi_n\frac{d\phi_m}{dx} - \phi_m\frac{d\phi_n}{dx}\right)\right)$$

$$(\lambda_n - \lambda_m)\int_a^b \sigma(x)\,\phi_n(x)\,\phi_m(x)\,dx = \int_a^b \frac{d}{dx}\left(p\left(\phi_n\frac{d\phi_m}{dx} - \phi_m\frac{d\phi_n}{dx}\right)\right)dx$$

$$(\lambda_n - \lambda_m)(\phi_n, \phi_m)_\sigma = \left(p(x)\left(\phi_n(x)\phi_m'(x) - \phi_m(x)\phi_n'(x)\right)\right)\Big|_a^b = 0$$

because ϕ_n and ϕ_m satisfy the boundary conditions this is zero

Therefore

$$(\lambda_n - \lambda_m)\,(\varphi_n, \varphi_m)_\sigma = 0$$

Since $\lambda_n \neq \lambda_m$ then $(\varphi_n, \varphi_m)_\sigma = 0$. $\boxed{\checkmark}$

1. All the eigenvalues λ are real.

Suppose λ were an eigenvalue that was not real.
Then $\bar{\lambda} \neq \lambda$.

Let φ be the eigenfunction for eigenvalue λ.
Thus $L\varphi = -\lambda \sigma \varphi$

Claim $\bar{\varphi}$ is an eigenfunction with eigenvalue $\bar{\lambda}$

Claim $L\bar{\varphi} = \overline{L\varphi}$

$$L\varphi = \frac{d}{dx}\left(p(x)\frac{d\varphi}{dx}\right) + q(x)\varphi$$

$$\overline{L\varphi} = \overline{\frac{d}{dx}\left(p(x)\frac{d\varphi}{dx}\right) + q(x)\varphi}$$

by hypothesis p and q are real valued

$\overline{p(x)} = p(x)$

$\overline{q(x)} = q(x)$

The real question is whether

$$\boxed{\overline{\frac{d}{dx}\varphi} = \frac{d}{dx}\overline{\varphi}}$$

need to know what differentiation of complex valued function even is to answer this...

Next time...