$$\underbrace{\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right)}_{L\varphi} + q(x)\phi + \lambda\sigma(x)\phi = 0, \qquad a < x < b,$$

where β_i are real. In addition, to be called regular, the coefficients p, q, and σ must be real and continuous everywhere (including the endpoints) and $p > 0$ and $\sigma > 0$ everywhere

$$\beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) = 0$$

$$\beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) = 0,$$

$$\beta_1\bar{\varphi}(a) + \beta_2\overline{\frac{d\varphi}{dx}}(a) = 0$$

$$\beta_3\bar{\varphi}(b) + \beta_4\overline{\frac{d\varphi}{dx}}(b) = 0$$

so $\bar{\varphi}$ also satisfies bndry cond...

1. All the eigenvalues λ are real.

Lagrange identity...

$$uLv - vLu = \frac{d}{dx}\left(p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right)$$

How does complex conjugation interact with derivatives

Claim: $\overline{\frac{d}{dx}\varphi} = \frac{d}{dx}\overline{\varphi}$

Let $\varphi: \mathbb{R} \to \mathbb{C}$ $\qquad \varphi(x) = f(x) + i\,g(x)$

where $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$.

$$\varphi'(x) = \lim_{h\to 0} \frac{\varphi(x+h) - \varphi(x)}{h} = \lim_{h\to 0} \frac{f(x+h) + ig(x+h) - (f(x) + ig(x))}{h}$$

$$= \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h} + \frac{ig(x+h) - ig(x)}{h}\right)$$

$$= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} + i\lim_{h\to 0}\frac{g(x+h) - g(x)}{h} = f'(x) + i\,g'(x)$$

$$\overline{\frac{d}{dx}\varphi(x)} = \overline{\varphi'(x)} = \overline{f'(x) + i g'(x)} = f'(x) - i g'(x)$$

$$\frac{d}{dx}\overline{\varphi(x)} = \bar{\varphi}'(x) = \lim_{h\to 0} \frac{\overline{\varphi(x+h)} - \overline{\varphi(x)}}{h} = \lim_{h\to 0} \frac{\overline{f(x+h) + ig(x+h) - (f(x) + ig(x))}}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - ig(x+h) - (f(x) - ig(x))}{h} = \lim_{h\to 0} \frac{f(x+h) - ig(x+h) - f(x) + ig(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x) - ig(x+h) + ig(x)}{h} = \lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h} - \frac{ig(x+h) - ig(x)}{h} \right)$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - i \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f'(x) - ig'(x)$$

Conclusion $\overline{\frac{d}{dx}\varphi(x)} = \frac{d}{dx}\overline{\varphi(x)}$

Claim $L\bar{\varphi} = \overline{L\varphi}$

$$\overline{L\varphi} = \overline{\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi} = \overline{\frac{d}{dx}\left(p(x)\frac{d\varphi}{dx}\right)} + \overline{q(x)\varphi}$$

$$= \frac{d}{dx}\overline{\left(p(x)\frac{d\varphi}{dx}\right)} + \overline{q(x)}\,\overline{\varphi} = \frac{d}{dx}\left(\overline{p(x)}\,\overline{\frac{d\varphi}{dx}}\right) + q(x)\overline{\varphi}$$

$$= \frac{d}{dx}\left(p(x)\frac{d\overline{\varphi}}{dx}\right) + q(x)\overline{\varphi} = L\bar{\varphi}$$

Let φ be an eigenfunct with eigenvalue λ.

Then $L\varphi = -\lambda\sigma(x)\varphi$. consequently

$$\overline{L\varphi} = \overline{-\lambda\sigma(x)\varphi}$$

Thus

$$L\bar{\varphi} = -\bar{\lambda}\,\overline{\sigma(x)}\,\bar{\varphi}$$

since σ is real

$$L\bar{\varphi} = -\bar{\lambda}\sigma(x)\bar{\varphi}$$

Lagrange identity... $u = \varphi$ and $v = \bar{\varphi}$

$$\boxed{uLv - vLu = \frac{d}{dx}\left(p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right)}$$

$$\varphi L\bar{\varphi} - \bar{\varphi}L\varphi = \frac{d}{dx}\left(p\left(\varphi\bar{\varphi}' - \bar{\varphi}\varphi'\right)\right)$$

$$-\varphi\bar{\lambda}\sigma(x)\bar{\varphi} + \bar{\varphi}\lambda\sigma(x)\varphi = \frac{d}{dx}\left(p\left(\varphi\bar{\varphi}' - \bar{\varphi}\varphi'\right)\right)$$

$$(-\bar{\lambda} + \lambda)\left(\sigma(x)\varphi(x)\bar{\varphi}(x)\right) = \frac{d}{dx}\left(p\left(\varphi\bar{\varphi}' - \bar{\varphi}\varphi'\right)\right)$$

$$(\lambda - \bar{\lambda})\int_a^b \sigma(x)\varphi(x)\bar{\varphi}(x)\,dx = p(x)\left(\varphi(x)\bar{\varphi}'(x) - \bar{\varphi}(x)\varphi'(x)\right)\Big|_a^b = 0$$

since both φ and $\bar{\varphi}$ satisfy the boundary cond.

Thus

$$(\lambda - \bar{\lambda})\int_a^b \sigma(x)\varphi(x)\bar{\varphi}(x)\,dx = 0$$

Either $\lambda - \bar{\lambda} = 0$ (which is what we want to show)

$$\text{or} \quad \int_a^b \sigma(x)\,\varphi(x)\,\bar{\varphi}(x)\,dx = 0 \qquad \text{(not this, see below)}$$

$$\varphi(x)\bar{\varphi}(x) = \big(f(x)+ig(x)\big)\big(f(x)-ig(x)\big) = f(x)^2 - i^2 g(x)^2 - \cancel{if(x)g(x)} + \cancel{if(x)g(x)}$$

$$\varphi(x)\bar{\varphi}(x) = f(x)^2 + g(x)^2 \geqslant 0$$

Since $\varphi(x)$ is not the zero function there is some point x_0 where either $f(x_0) \neq 0$ or $g(x_0) \neq 0$. Thus $\varphi(x_0)\bar{\varphi}(x_0) > 0$

Since φ is differentiable it is continuous.

So there is $\varepsilon > 0$ and $\delta > 0$ such that

$$\varphi(x)\bar{\varphi}(x) \geqslant \varepsilon \quad \text{for all} \quad x \in [x_0-\delta,\, x_0+\delta].$$

$$\int_a^b \sigma(x)\,\varphi(x)\bar{\varphi}(x)\,dx \geqslant \int_{x_0-\delta}^{x_0+\delta} \sigma(x)\,\varphi(x)\bar{\varphi}(x)\,dx \geqslant \int_{x_0-\delta}^{x_0+\delta} \sigma(x)\,\varepsilon\,dx$$

$$\text{let } m = \min\{\sigma(x) : x \in [x_0-\delta,\, x_0+\delta]\}.$$

$$\int_a^b \sigma(x)\,\varphi(x)\bar{\varphi}(x)\,dx \geqslant \int_{x_0-\delta}^{x_0+\delta} m\varepsilon\,dx = 2\delta m\varepsilon > 0$$