6. Any eigenvalue can be related to its eigenfunction by the **Rayleigh quotient:**

$$\lambda = \frac{-p\phi\, d\phi/dx\big|_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2]\, dx}{\int_a^b \phi^2 \sigma\, dx},$$

where the boundary conditions may somewhat simplify this expression.

This is related to a least squares approximation of the eigenvalue.

Consider the least squares problem find the y that

minimizes $\|By - c\|$ (ie. solve $By = c$ by least squares)

where $B \in \mathbb{R}^{m \times n}$ with $m > n$ and independent columns.

rows cols

Solution is solve the normal equations for y.

$$\underbrace{B^T B}_{\text{this is invertible because columns are independent}} y = B^T c \qquad \underset{n\times m}{B^T}\ \underset{m\times n}{B} \in \mathbb{R}^{n\times n}$$

$$y = (B^T B)^{-1} B^T c$$

Estimate eigenvalue using least squares

$$Ax = \lambda x$$

find the λ that minimizes $\|Ax - \lambda x\| = \|x\lambda - Ax\|$

Solution is to solve normal equations

$$x^T x \lambda = x^T A x$$

$$(x \cdot x)\lambda = x \cdot Ax$$

The better x approximates the eigenvector, the better λ approximates the eigenvalue.

$$\boxed{\lambda = \frac{x \cdot Ax}{x \cdot x}}$$

Rayleigh quotient: (lin. alg).

Now the same thing for the eigenvalue-eigenfunction problem

$$L\phi = -\lambda\sigma\phi \quad \text{or} \quad L\phi(x) = -\lambda\sigma(x)\phi(x)$$

So the same this.

$$\int_a^b \phi(x) L\phi(x)\,dx = \int_a^b -\lambda\sigma(x)\phi(x)\phi(x)\,dx$$

$$(\phi, L\phi) = -\lambda(\phi,\phi)_\sigma \qquad \boxed{\lambda = \frac{-(\phi, L\phi)}{(\phi,\phi)_\sigma}}$$

Rayleigh quotient: (Sturm-Liouville)

$$\lambda = \frac{-p\phi\, d\phi/dx\big|_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2]\,dx}{\int_a^b \phi^2\sigma\,dx},$$

$$\boxed{\underbrace{\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi}_{L\phi(x)} + \lambda\sigma(x)\phi = 0,}$$

$$L = \frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi$$

$$(\phi, L\phi) = \int_a^b \phi(x)\left(\frac{d}{dx}\left(p(x)\phi'(x)\right) + q(x)\phi(x)\right)dx$$

$$= \int_a^b \left\{\phi(x)\frac{d}{dx}\left(p(x)\phi'(x)\right) + q(x)\phi(x)^2\right\}dx$$

$u = \varphi(x)$ $\qquad du = \varphi'(x)\,dx$

$dv = \frac{d}{dx}\left(p(x)\,\varphi'(x)\right)dx$ $\qquad v = p(x)\,\varphi'(x)$

$$\int_a^b \varphi(x)\,\frac{d}{dx}\left(p(x)\,\varphi'(x)\right) = p(x)\varphi(x)\varphi'(x)\Big|_a^b - \int_a^b p(x)\,\varphi'(x)^2\,dx$$

$$(\varphi, L\varphi) = p(x)\varphi(x)\varphi'(x)\Big|_a^b - \int_a^b \left\{p(x)\,\varphi'(x)^2 - q(x)\,\varphi(x)^2\right\}dx$$

$$\lambda = \frac{-p\phi\, d\phi/dx\big|_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2]\,dx}{\int_a^b \phi^2\sigma\,dx},$$

$$\lambda = \frac{-(\varphi, L\varphi)}{(\varphi,\varphi)_\sigma} = \frac{-p(x)\varphi(x)\varphi'(x)\Big|_a^b + \int_a^b \left\{p(x)\,\varphi'(x)^2 - q(x)\,\varphi(x)^2\right\}dx}{\int_a^b \varphi^2(x)\,\sigma(x)\,dx}$$

The better x approximates the eigenfunction, the better λ approximates the eigenvalue..

Note if you plug in arbitrary differentiable functions that can estimate a range of possible eigenvalues....

This term vanishes when boundary cond are used.

• This idea is related to exercises 5.6.1 and 5.6.2

see the example on page 186.

Let λ be an eigenvalue and φ_1 be an eigenfunction so that $\quad L\varphi_1 = -\lambda\sigma\varphi_1$.

Suppose φ_2 is another eigenfunction so $L\varphi_2 = -\lambda\sigma\varphi_2$

Then $\phi_1 = c\,\phi_2$.

$$\beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) = 0$$

$$\beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) = 0,$$

using these boundary conditions

Given λ solve this ODE with

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi = -\lambda\sigma\phi$$

Finish next time...