Eigenfunctions corresponding to the same eigenvalue are scalar multiples of each other.

$$\beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) = 0$$
$$\beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) = 0,$$

need these boundary condition.

for uniqueness (up to scalar multiples) of solutions to the ODE

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi = -\lambda\sigma\phi$$

Lagrange identity:

$$uLv - vLu = \frac{d}{dx}\left(p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right)$$

Now, suppose

$$L\phi_1 = -\lambda\sigma\phi_1 \quad \text{and} \quad L\phi_2 = -\lambda\sigma\phi_2$$

Plug into the Lagrange identity:

$$\phi_1 L\phi_2 - \phi_2 L\phi_1 = \frac{d}{dx}\left(p\left(\phi_1\frac{d\phi_2}{dx} - \phi_2\frac{d\phi_1}{dx}\right)\right)$$

$$\phi_1 L\phi_2 - \phi_2 L\phi_1 = \phi_1(\cancel{-\lambda\sigma\phi_2}) - \phi_2(\cancel{-\lambda\sigma\phi_1}) = 0$$

Thus

$$\frac{d}{dx}\left(p\left(\phi_1\frac{d\phi_2}{dx} - \phi_2\frac{d\phi_1}{dx}\right)\right) = 0$$

so

$$p\left(\phi_1\frac{d\phi_2}{dx} - \phi_2\frac{d\phi_1}{dx}\right) = \text{const}$$

Use the boundary conditions to find the constant.

$$p\left(\phi_1 \frac{d\phi_2}{dx} - \phi_2 \frac{d\phi_1}{dx}\right)\Bigg|_{x=a} = p(a)\left(\phi_1(a)\left(-\frac{\beta_1}{\beta_2}\phi_2(a)\right) - \phi_2(a)\left(-\frac{\beta_1}{\beta_2}\phi_1(a)\right)\right) = 0$$

$$\beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) = 0$$

Case $\beta_2 \neq 0$. Then $\frac{d\phi_1}{dx}(a) = -\frac{\beta_1}{\beta_2}\phi_1(a)$ and $\frac{d\phi_2}{dx}(a) = -\frac{\beta_1}{\beta_2}\phi_2(a)$

thus const $= 0$ and

$$\cancel{p}\left(\phi_1 \frac{d\phi_2}{dx} - \phi_2 \frac{d\phi_1}{dx}\right) = 0$$

where $\beta_i$ are real. In addition, to be called regular, the coefficients $p$, $q$, and $\sigma$ must be real and continuous everywhere (including the endpoints) and $\boxed{p > 0}$ and $\sigma > 0$ everywhere (also including the endpoints). For the regular Sturm–Liouville eigenvalue problem, many

$$\phi_1 \frac{d\phi_2}{dx} - \phi_2 \frac{d\phi_1}{dx} = 0$$

The book considers the derivative of $\phi_1/\phi_2$.

$$\frac{d}{dx}\left(\frac{\phi_1}{\phi_2}\right) = \frac{\phi_1'\phi_2 - \phi_1\phi_2'}{\phi_2^2} = \frac{0}{\phi_2^2} = 0$$

note be careful because $\phi_2$ is sometimes $0$.

Therefore $\frac{\phi_1}{\phi_2} = c$ and so $\phi_1 = c\phi_2$.

note since $\phi_2$ is a solution to a $2^{nd}$ order differential equation then $\phi_2'$ is continuous. this and the fact that the zeros of $\phi_2$ are isolated

$\phi_n(x)$ has exactly $n-1$ zeros for $a < x < b$.

implies on each interval where $\phi_2(x) \neq 0$ that the same constant $c$ is there...

Use of the Rayliegh quotient to approximate the smallest eigenvalue in a Sturm–Liouville problem.

Work this for linear algebra first...

Let $A \in \mathbb{R}^{n \times n}$ with $A = A^T$. Therefore the spectral theorem implies there is an orthonormal basis of eigenvectors with real eigenvalues. Thus

$$Ax_i = \lambda_i x_i \quad \text{for } i = 1, \dots, n$$

$$\lambda_i \in \mathbb{R}$$

$$x_i \cdot x_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

The Rayliegh quotient is

$$RQ(x) = \frac{x \cdot Ax}{x \cdot x} \approx \lambda_i \quad \text{if } x \text{ approximates } x_i$$

Note

$$RQ(x_i) = \frac{x_i \cdot Ax_i}{x_i \cdot x_i} = \frac{x_i \cdot \lambda_i x_i}{x_i \cdot x_i} = \lambda_i \frac{x_i \cdot x_i}{x_i \cdot x_i} = \lambda_i$$

Suppose $x$ is not an eigenvector. Since $x_i$ form a basis

$$x = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

$$RQ(x) = \frac{(c_1 x_1 + c_2 x_2 + \dots + c_n x_n) \cdot A (c_1 x_1 + c_2 x_2 + \dots + c_n x_n)}{(c_1 x_1 + c_2 x_2 + \dots + c_n x_n) \cdot (c_1 x_1 + c_2 x_2 + \dots + c_n x_n)}$$

$$= \frac{(c_1 x_1 + c_2 x_2 + \dots + c_n x_n) \cdot (c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + \dots + c_n \lambda_n x_n)}{(c_1 x_1 + c_2 x_2 + \dots + c_n x_n) \cdot (c_1 x_1 + c_2 x_2 + \dots + c_n x_n)}$$

$$= \frac{c_1^2 \lambda_1 + c_2^2 \lambda_2 + \dots + c_n^2 \lambda_n}{c_1^2 + c_2^2 + \dots + c_n^2} \quad \Big\} \text{ weighted average of the } \lambda_i\text{'s}$$

$$\min\{\lambda_i\} \le \frac{c_1^2\lambda_1 + c_2^2\lambda_2 + \dots + c_n^2\lambda_n}{c_1^2 + c_2^2 + \dots + c_n^2} \le \max\{\lambda_i\}$$

If we order the eigenvalues as $\lambda_1 < \lambda_2 < \ldots < \lambda_n$ then

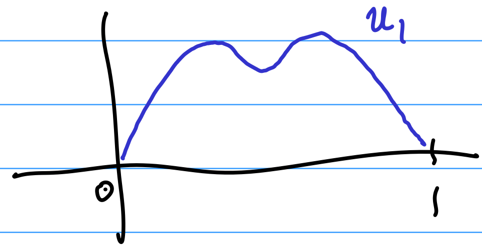
$$\lambda_1 \le RQ(x) \le \lambda_n.$$

For the sturm-Liouville problem.

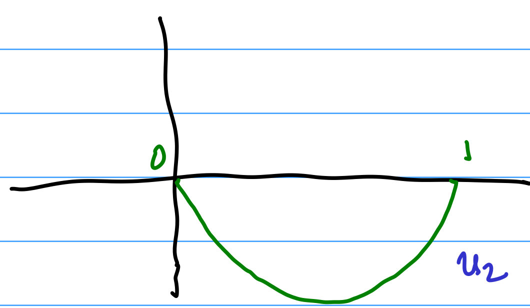
$$\lambda_1 \le \underbrace{\frac{-p\phi\, d\phi/dx\big|_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2]\, dx}{\int_a^b \phi^2\sigma\, dx}}_{\text{upper bound on } \lambda_1},$$

Example: find an upper bound on the least eigenvalue in the problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0$$
$$\phi(0) = 0$$
$$\phi(1) = 0.$$

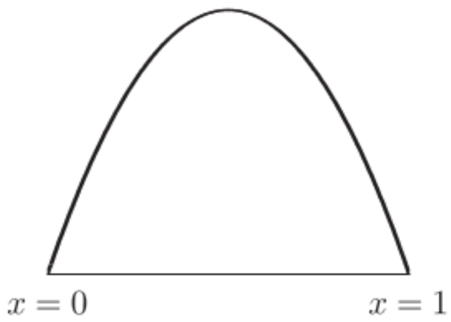



$\lambda_1 \le RQ(u_1)$




$\lambda_1 \le RQ(u_2)$

$\lambda \le \min\{RQ(u) : u$ is a function that satisfies the boundary cond$\}$




$u_T = x - x^2$

$$RQ(u_T) = \frac{\cancel{-p\phi\, d\phi/dx\big|_a^b} + \int_a^b [p(d\phi/dx)^2 - q\phi^2]\, dx}{\int_a^b \phi^2 \sigma\, dx}$$

$\phi = u_T$

$$= \frac{\int_0^1 \left(\frac{d u_T}{dx}\right)^2 dx}{\int_0^1 (u_T)^2\, dx}$$

$$= \frac{\int_0^1 (1-2x)^2\, dx}{\int_0^1 (x-x^2)^2\, dx} = \frac{\int_0^1 (1-4x+4x^2)\, dx}{\int_0^1 (x^2-2x^3+x^4)\, dx} = \frac{1-2+\frac{4}{3}}{\frac{1}{3}-\frac{1}{2}+\frac{1}{5}} = 10,$$

Thus $\lambda_1 \lesssim 10$.

Remark $\lambda_1 \approx \pi^2$