motivation and intuition

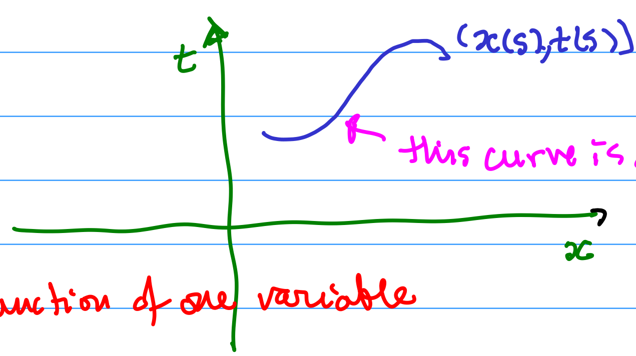
Technique

HW from here

Consider a first order PDE solve for $w(x,t)$

$$\frac{\partial w}{\partial t} + t\frac{\partial w}{\partial x} = 1 \qquad w(x_0, 0) = f(x_0)$$

Treat $t = t(s)$ and $x = x(s)$ where s is another variable.



this curve is called a characteristic for the PDE provided...

just a function of one variable

$$w(s) = w(x(s), t(s))$$

$$t\frac{\partial w}{\partial x} + 1\frac{\partial w}{\partial t} = 1$$

compare

$$\frac{dw}{ds} = \frac{\partial w}{\partial x}x'(s) + \frac{\partial w}{\partial t}t'(s)$$

This leads to a system of ODEs:

3 ODEs are equivalent to the PDE

$$\begin{cases} x'(s) = t \\ t'(s) = 1 \\ \frac{dw}{ds} = 1 \end{cases}$$

if these hold, then $\frac{dw}{ds}$ is exactly the left side of the PDE

Solve them

constant of integration

① $t'(s) = 1$ $\qquad t(s) = s + t_0$

constant of integration

② $x'(s) = t = s + t_0$ $\qquad x(s) = \frac{1}{2}s^2 + st_0 + x_0$

③ $\frac{dw}{ds} = 1$ $\qquad w(s) = s + w_0$

$w(s) = w(x(s), t(s))$

more clearly $\qquad w(x(s), t(s)) = s + w(x(0), t(0))$

substituting $\qquad w(\underbrace{\tfrac{1}{2}s^2 + st_0 + x_0}_{x}, \underbrace{s + t_0}_{t}) = s + w(x_0, t_0)$

solve for x and t in terms of the other variables

$t = s + t_0 \qquad s = t - t_0$

$x = \frac{1}{2}s^2 + st_0 + x_0 \qquad x_0 = x - \frac{1}{2}s^2 - st_0$

$= x - \frac{1}{2}(t - t_0)^2 - (t - t_0)t_0$

Thus

$$w\left(\underbrace{\tfrac{1}{2}s^2+st_0+x_0}_{x},\ \underbrace{s+t_0}_{t}\right) = s + w(x_0, t_0)$$

$$w(x,t) = t - t_0 + w\left(x - \tfrac{1}{2}(t-t_0)^2 - (t-t_0)t_0\ ,\ t_0\right)$$

In order to substitute the initial condition $w(y,0) = f(y)$

I'll take $t_0 = 0$ then

$$w(x,t) = t + w(x - \tfrac{1}{2}t^2, 0) = t + f(x - \tfrac{1}{2}t^2)$$

Solution $w(x,t) = t + f(x - \tfrac{1}{2}t^2)$.

$$1\,\frac{\partial w}{\partial t} + c\,\frac{\partial w}{\partial x} = e^{2x} \qquad w(x,0) = f(x)$$

Suppose $w(s) = w(x(s), t(s))$ then

$$\frac{dw}{ds} = \frac{\partial w}{\partial x}x'(s) + \frac{\partial w}{\partial t}t'(s) = e^{2x(s)}$$

provided

$$\begin{cases} x'(s) = c \\ t'(s) = 1 \end{cases}$$

3 ODEs

$$t(s) = \int 1\,ds = s + t_0$$

$$x(s) = \int c\,ds = cs + x_0$$

$$\frac{dw}{ds} = e^{2(cs+x_0)}$$

$$w(s) = \int e^{2(cs+x_0)}\,ds = \frac{1}{2c}e^{2(cs+x_0)} + \text{Const}$$

$$w(0) = \frac{1}{2c}e^{2(cs+x_0)} = \frac{1}{2c}e^{2x_0} + \text{Const}$$

$$\text{const} = w(0) - \frac{1}{2c}e^{2x_0}$$

$$w(s) = w(x(s), t(s)) = w(cs + x_0, s + t_0) = \frac{1}{2c} e^{2(cs + x_0)} + w(0) - \frac{1}{2c} e^{2x_0}$$

$$w(0) = w(x(0), t(0)) = w(x_0, t_0) =$$

Set $t_0 = 0$ so $w(0) = w(x_0, 0) = f(x_0)$

$w(x, 0) = f(x)$

Same

$$w(\underbrace{cs + x_0}_{x}, \underbrace{s}_{t}) = \frac{1}{2c} e^{2(cs + x_0)} + f(x_0) - \frac{1}{2c} e^{2x_0}$$

$x = cs + x_0$ $\quad$ $x_0 = x - cs = x - ct$

$t = s$

$$w(x, t) = \frac{1}{2c} e^{2(\cancel{ct} + x - \cancel{ct})} + f(x - ct) - \frac{1}{2c} e^{2(x - ct)}$$

$$w(x, t) = \frac{1}{2c} e^{2x} - \frac{1}{2c} e^{2(x - ct)} + f(x - ct)$$

$$\frac{\partial w}{\partial t} + 3t \frac{\partial w}{\partial x} = w \qquad w(x_0, 0) = f(x)$$

$$\frac{dw}{ds} = \frac{\partial w}{\partial x} x' + \frac{\partial w}{\partial t} t' = w$$

$x' = 3t$

$\boxed{t' = 1}$ $\quad$ $t = s + t_0 = s$ $\quad$ $t_0 = 0$

finish next time.