

12 The Method of Characteristics for Linear and Quasilinear

Wave Equations

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motivation and intuition

Technique

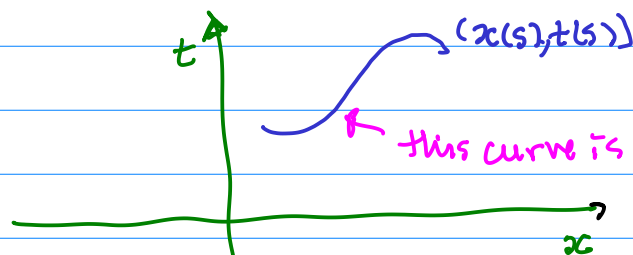
His from here

Consider a first order PDE solve for $w(x, t)$

$$\frac{\partial w}{\partial t} + t \frac{\partial w}{\partial x} = 1$$

$$w(x_0, 0) = f(x_0)$$

Treat $t = t(s)$ and $x = x(s)$ where s is another variable.



this curve is called a characteristic for the PDE provided ...

just a function of one variable

$$w(s) = w(x(s), t(s))$$

$$\frac{dw}{ds} = \frac{\partial w}{\partial x} x'(s) + \frac{\partial w}{\partial t} t'(s)$$

$$t \frac{\partial w}{\partial x} + 1 \frac{\partial w}{\partial t} = 1$$

compare

This leads to a system of ODEs.

3 ODEs
are equivalent
to the PDE

$$\begin{cases} x'(s) = t \\ t'(s) = 1 \\ \frac{dw}{ds} = 1 \end{cases}$$

if these hold, then $\frac{dw}{ds}$ is exactly the left side of the PDE

Solve them

①

$$t'(s) = 1$$

$$t(s) = s + t_0$$

constant of integration

②

$$x'(s) = t = s + t_0$$

$$x(s) = \frac{1}{2}s^2 + s t_0 + x_0$$

constant of integration

③

$$\frac{dw}{ds} = 1$$

$$w(s) = s + w_0$$

$$w(s) = w(x(s), t(s))$$

more clearly

$$w(x(s), t(s)) = s + w(x_0, t_0)$$

substituting

$$w\left(\underbrace{\frac{1}{2}s^2 + s t_0 + x_0}_x, \underbrace{s + t_0}_t\right) = s + w(x_0, t_0)$$

solve for x and t in terms of the other variables

$$t = s + t_0$$

$$s = t - t_0$$

$$x = \frac{1}{2}s^2 + s t_0 + x_0$$

$$x_0 = x - \frac{1}{2}s^2 - s t_0$$

$$= x - \frac{1}{2}(t - t_0)^2 - (t - t_0)t_0$$

Thus

$$w(\underbrace{\frac{1}{2}s^2 + st_0 + x_0}_x, \underbrace{s+t_0}_t) = s + w(x_0, t_0)$$

$$w(x, t) = t - t_0 + w(x - \frac{1}{2}(t-t_0)^2 - (t-t_0)t_0, t_0)$$

In order to substitute the initial condition $w(y, 0) = f(y)$

I'll take $t_0 = 0$ then

$$w(x, t) = t + w(x - \frac{1}{2}t^2, 0) = t + f(x - \frac{1}{2}t^2)$$

Solution $w(x, t) = t + f(x - \frac{1}{2}t^2)$.

$$1 \frac{\partial w}{\partial t} + c \frac{\partial w}{\partial x} = e^{2x}$$

$$w(x, 0) = f(x)$$

Suppose $w(s) = w(x(s), t(s))$ then

$$\frac{dw}{ds} = \frac{\partial w}{\partial x} x'(s) + \frac{\partial w}{\partial t} t'(s) = e^{2x(s)}$$

provided

$$\begin{cases} x'(s) = c \\ t'(s) = 1 \end{cases}$$

3 ODEs

$$t(s) = \int 1 ds = s + t_0$$

$$x(s) = \int c ds = cs + x_0$$

$$\frac{dw}{ds} = e^{2(cs+x_0)}$$

$$w(s) = \int e^{2(cs+x_0)} ds = \frac{1}{2c} e^{2(cs+x_0)} + \text{const}$$

$$w(0) = \frac{1}{2c} e^{2x_0} = \frac{1}{2c} e^{2x_0} + \text{const}$$

$$\text{const} = w(0) - \frac{1}{2c} e^{2x_0}$$

$$w(s) = w(x(s), t(s)) = w(cs + x_0, s + t_0) = \frac{1}{2c} e^{2(cs + x_0)} + w(0) - \frac{1}{2c} e^{2x_0}$$

$$w(0) = w(x(0), t(0)) = w(x_0, t_0) =$$

$$\text{Set } t_0 = 0 \text{ so } w(0) = w(x_0, 0) = f(x_0)$$

$$w(x, 0) = f(x)$$

same

$$w(cs + x_0, s) = \frac{1}{2c} e^{2(cs + x_0)} + f(x_0) - \frac{1}{2c} e^{2x_0}$$

$$x = cs + x_0$$

$$t = s$$

$$x_0 = x - cs = x - ct$$

$$w(x, t) = \frac{1}{2c} e^{2(ct + x - ct)} + f(x - ct) - \frac{1}{2c} e^{2(x - ct)}$$

$$w(x, t) = \frac{1}{2c} e^{2x} - \frac{1}{2c} e^{2(x - ct)} + f(x - ct)$$

$$\frac{\partial w}{\partial t} + 3t \frac{\partial w}{\partial x} = w$$

$$w(x_0, 0) = f(x)$$

$$\frac{dw}{ds} = \frac{\partial w}{\partial x} x' + \frac{\partial w}{\partial t} t' = w$$

$$x' = 3t$$

$$t' = 1$$

$$t = s + t_0 = s$$

$$t_0 = 0$$

finish next time.