Abstractly: $u = u(x,t)$

PDE: $\boxed{a(x,t)\frac{\partial u}{\partial x} + b(x,t)\frac{\partial u}{\partial t}} = c(x,t)u + d(x,t)$

Idea is introduce a parameter s and a function dependency

$x = x(s), \quad t = t(s)$

Thus

$u(s) = u(x(s), t(s))$

$a(s) = a(x(s), t(s))$

$b(s) = b(x(s), t(s))$

$c(s) = c(x(s), t(s))$

$d(s) = d(x(s), t(s))$

Use chain rule to create a system of ODEs...

$\frac{du}{ds} = \boxed{\frac{\partial u}{\partial x}x'(s) + \frac{\partial u}{\partial t}t'(s)}$

(want these the same)

Two ODEs

$x'(s) = a(x(s), t(s))$

$t'(s) = b(x(s), t(s))$

If they are the same then - if the right hand sides are also the same then this corresponds to the same equation. Gives us the tranformed PDE along the characteristics

one more ODE

$$\frac{du}{ds} = c(x(s), t(s))u(x(s), t(s)) + d(x(s), t(s))$$

Solving the 3 ODEs gives a solution to the PDE.

Example $1\frac{\partial u}{\partial x} + 1\frac{\partial u}{\partial t} = -2u$ such that $u(x,0) = \sin(x)$.

Write $x = x(s)$ and $t = t(s)$ then

$$\frac{du}{ds} = \frac{\partial u}{\partial x}x'(s) + \frac{\partial u}{\partial t}t'(s)$$

Along the characteristics

$$\frac{du}{ds} = -2u$$

Two ODEs

$x'(s) = 1 \qquad x(s) = s + c_1$

$t'(s) = 1 \qquad t(s) = s + c_2$

Change of variables for x, t in terms of s, c_1 and c_2.

Three ODEs

$$\frac{du}{ds} = -2u \quad , \qquad x'(s) = 1 \quad , \qquad t'(s) = 1$$

$$x(s) = s + c_1 \qquad t(s) = s + c_2$$

lets say $s=0$ corresponds to the initial cond.

$$u(x_0, 0) = \sin(x_0)$$

$t(0) = 0 + c_2 = t_0 = 0$ so $c_2 = 0$

$x(0) = 0 + c_1 = x_0$ so $c_1 = x_0$

$$x(s) = s + x_0 \qquad t(s) = s$$

Now solve this ODE:

$$\frac{du}{ds} = -2u \qquad u(s) = c_3 e^{-2s}$$

integrating factor method

$$\frac{du}{ds} + 2u = 0$$

$$\mu = e^{2s} \qquad \frac{d\mu}{ds} = 2e^{2s}$$

$$e^{2s}\left(\frac{du}{ds} + 2u\right) = e^{2s}\,0$$

$$\underbrace{e^{2s}\frac{du}{ds} + 2e^{2s}u}_{} = 0$$

$$\frac{d}{ds}\left(e^{2s}u\right) = 0 \qquad e^{2s}u = c_3 \qquad \text{thus} \quad u = c_3 e^{-2s}$$

Solve for c_3...

$$u(x(s), t(s)) = c_3 e^{-2s} \qquad x(s) = s + x_0 \qquad t(s) = s$$

$$u(s+x_0, s) = c_3 e^{-2s}$$

$$u(x_0, 0) = c_3 e^{-2\cdot 0} = c_3 = \sin(x_0)$$ by the initial condition

Solution

$$u(s+x_0, s) = \sin(x_0)\, e^{-2s}$$

Solve for $u(x,t)$ so $x = s + x_0$ and $t = s$

$s = t$

$$x = t + x_0$$

$$x_0 = x - t$$

$$\boxed{u(x,t) = \sin(x-t)\, e^{-2t}}$$

Example: $x \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial t} = -tu$ $\quad u(x,0) = f(x)$.

let $x = x(s)$ and $t = t(s)$ then by Chain rule

$$\frac{du}{ds} = \frac{\partial u}{\partial x} x'(s) + \frac{\partial u}{\partial t} t'(s)$$

implies that along the characteristics given by

$$x'(s) = x(s) \qquad t'(s) = 1$$

from before $t = s + c_2$

so $t = s$

$t=0$ time at initial condition

we have

$$\frac{du}{ds} = -tu$$

Now solve $x' = x$. $\quad x = c_1 e^s = x_0 e^s$

the value of x at $s=0$.

$$\frac{du}{ds} = -su \qquad \frac{du}{ds} + su = 0$$ solve by integrating factor

$$\mu = e^{\int s\,ds} = e^{\frac{1}{2}s^2}$$

$$e^{\frac{1}{2}s^2}\left(\frac{du}{ds}+su\right)=0 \quad \text{so} \quad \underbrace{e^{\frac{1}{2}s^2}\frac{du}{ds} + se^{\frac{1}{2}s^2}u}_{\frac{d}{ds}\left(e^{\frac{1}{2}s^2}u\right)} = 0$$

$$\frac{d}{ds}\left(e^{\frac{1}{2}s^2}u\right)=0$$

$$e^{\frac{1}{2}s^2}u = c_3$$

$$= c_3 e^{-\frac{1}{2}s^2}$$

Solve for c_3

$$u(x(s), t(s)) = c_3 e^{-\frac{1}{2}s^2} \qquad t = s \qquad x = x_0 e^{s}$$

$$u(x_0 e^{s}, s) = c_3 e^{-\frac{1}{2}s^2}$$

since $u(x,0) = f(x)$

plug in $s=0$

$$f(x_0) = u(x_0, 0) = c_3 e^{0} = c_3$$

thus $u(\underbrace{x_0 e^{s}}_{x}, \underbrace{s}_{t}) = f(x_0)\, e^{-\frac{1}{2}s^2}$

Now solve for $u(x,t)$

$$x = x_0 e^{s} \qquad t = s$$

$$x_0 = x e^{-s} = x e^{-t}$$

The solution is

$$\boxed{u(x,t) = f(x e^{-t})\, e^{-\frac{1}{2}t^2}}$$

Example: $x\dfrac{\partial u}{\partial x} + t\dfrac{\partial u}{\partial t} = -2u$, $\quad u(x,1) = \sin x$

$x_0 \qquad x_0$

Characteristic equations

$$x'(s) = x(s) \qquad t'(s) = t(s)$$

$$x(s) = c_1 e^s \qquad t(s) = c_2 e^s$$

$$x(s) = x_0 e^s \quad \text{so that} \quad x(0) = x_0$$

$$t(s) = e^s \quad \text{so that} \quad t(0) = 1$$

Along the characteristics

$$\frac{du}{ds} = -2u \quad \text{so} \quad u = c_3 e^{-2s}$$

Thus

$$u(x_0 e^s, e^s) = c_3 e^{-2s}$$

$u(x,1) = \sin x$

$x_0 \qquad x_0$

Set $s=0$

$$u(x_0, 1) = c_3 \quad \text{so} \quad c_3 = \sin(x_0)$$

Thus

$$u(x_0 e^s, e^s) = \sin(x_0)\, e^{-2s}$$

What's left is to solve for $u(x,t)$.

$$x = x_0 e^s \qquad t = e^s$$

$$x = x_0 t \qquad s = \ln t$$

$$x_0 = x/t$$

$$u(x,t) = \sin(x/t)\, e^{-2\ln t}$$

Solution $\boxed{u(x,t) = \sin(x/t)\, \dfrac{1}{t^2}}$