$$1\frac{\partial \rho}{\partial t} + t\frac{\partial \rho}{\partial x} = 5 \qquad \rho(x,0) = f(x)$$

might as well paraterize the characteristics by t

But let's do the whole thing

$$\rho(s) = \rho(x(s), t(s)) \qquad t = t(s) \qquad x = x(s)$$

$$\frac{d\rho}{ds} = \frac{\partial \rho}{\partial x} x'(s) + \frac{\partial \rho}{\partial t} t'(s)$$

The ODEs governing the characteristics are

so $x(0) = x_0$

$$x'(s) = t = s \qquad x(s) = \int s\,ds = \frac{1}{2}s^2 + c_2 = \frac{1}{2}s^2 + x_0$$

$$t'(s) = 1 \qquad t(s) = s + c_1$$

Substitute $t = s$

set constant equal $c_1 = 0$ so that $s = 0$ corresponds to the initial condition $\rho(x,0) = f(x)$

$t(0) = 0$

$$\frac{d\rho}{ds} = 5 \qquad \rho(s) = 5s + c_3$$

$$\rho\left(\tfrac{1}{2}s^2 + x_0, s\right) = \rho(x(s), t(s)) = 5s + c_3$$

set $s=0$ $\quad \rho(x_0, 0) = 5\cdot 0 + c_3 \quad c_3 = \rho(x_0, 0) = f(x_0)$

Therefore

$$\boxed{\rho\left(\tfrac{1}{2}s^2 + x_0, s\right) = 5s + f(x_0)}$$

implicit solution

Solve for $\rho(x,t)$

$x = \frac{1}{2}s^2 + x_0 \qquad t = s$

$$x = \frac{1}{2}t^2 + x_0$$

$$x_0 = x - \frac{1}{2}t^2$$

$$\rho(x,t) = 5t + f\left(x_0 - \tfrac{1}{2}t^2\right)$$

$\frac{1}{2}$ disappears if one makes the error not substituting for t here

Traffic flow $\quad 1\dfrac{\partial \rho}{\partial t} + c(\rho)\dfrac{\partial \rho}{\partial x} = 0$ with $\boxed{\rho(x,0) = f(x)}$

So use t for the parameterization $x = x(t)$, $t = t$

$$\rho = \rho(x(t), t)$$

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial x} x'(t) + 1\frac{\partial \rho}{\partial t} = 0$$

ODEs $\quad x'(t) = c(\rho) \qquad \dfrac{d\rho}{dt} = 0 \qquad \rho(x(t),t) = \text{constant}$

$\rho(x(0), 0) = f(x(0))$

$$x'(t) = c\big(f(x(0))\big)$$

$$x(t) = c\big(f(x(0))\big)t + x(0)$$

$$x(t) = c\big(f(x_0)\big)t + x_0$$

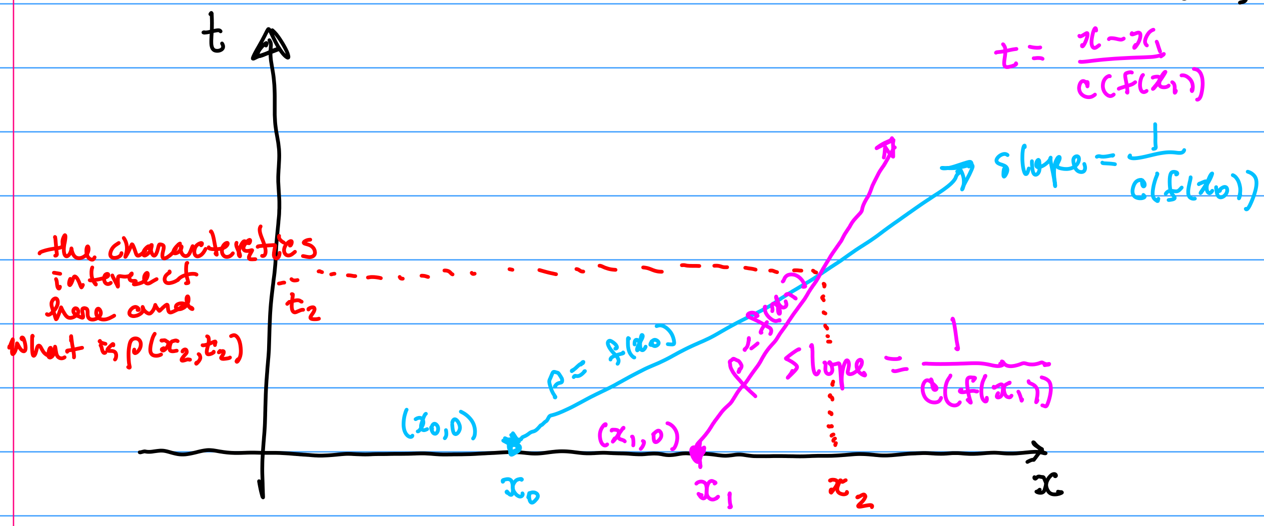
Solution

$$\rho\big(c(f(x_0))t + x_0, t\big) = f(x_0) \qquad \text{implicit}$$

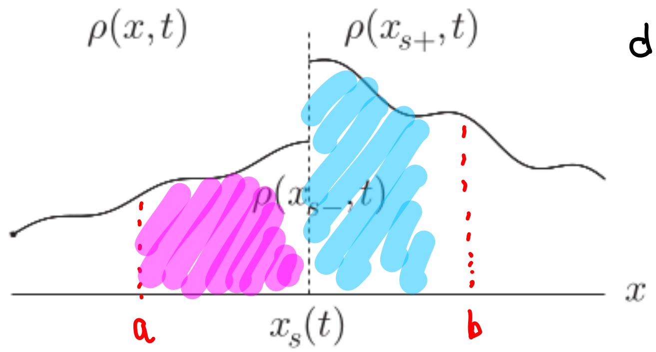
$$\rho(x,t) = f(x_0) \quad \text{where} \quad x = c(f(x_0))t + x_0$$

Need to know what f is in order to solve for x_0 in terms of x, and it might not be possible...

Graph characteristics $x = c(f(x_0))t + x_0$ or $t = \dfrac{x - x_0}{c(f(x_0))}$



Remark as $t \to t_2^-$ then the difference between the location of the density $f(x_0)$ cars and the density $f(x_1)$ cars becomes closer and closer... then there is a shock at $t = t_2$ and the solution has a jump discontinuity.



differential equation applies on the pieces separately.

$\int_a^b \rho(x,t)\,dx$ total number of cars between a and b at time t.

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx = q(a,t) - q(b,t)$$

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx = \frac{d}{dt}\int_a^{x_s(t)} \rho(x,t)\,dx + \frac{d}{dt}\int_{x_s(t)}^b \rho(x,t)\,dx$$

$$\frac{d}{dt}\int_a^{x_s(t)} \rho(x,t)\,dx = \frac{d}{dz}\int_a^{z} \rho(x,t)\,dx \Big|_{z = x_s(t)^-} x_s'(t) + \int_a^{x_s(t)} \frac{\partial \rho}{\partial t}(x,t)\,dx$$

$$= \rho(x_s(t)^-, t)\, x_s'(t) + \int_a^{x_s(t)} \frac{\partial \rho}{\partial t}(x,t)\,dx$$

$$\frac{d}{dt}\int_{x_s(t)}^b \rho(x,t)\,dx = \frac{d}{dz}\int_z^{b} \rho(x,t)\,dx \Big|_{z = x_s(t)^+} x_s'(t) + \int_{x_s(t)}^b \frac{\partial \rho}{\partial t}(x,t)\,dx$$

$$= -\rho(x_s(t)^+, t)\, x_s'(t) + \int_{x_s(t)}^b \frac{\partial \rho}{\partial t}(x,t)\,dx$$

$$\rho(x_s(t)^-, t)\, x_s'(t) + \int_a^{x_s(t)} \frac{\partial \rho}{\partial t}(x,t)\,dx - \rho(x_s(t)^+, t)\, x_s'(t) + \int_{x_s(t)}^{b} \frac{\partial \rho}{\partial t}(x,t)\,dx = q(a,t) - q(b,t)$$

- Solve for $x_s'(t)$ to see how the shock moves on the separate intervals $(a, x_s(t))$ and $(x_s(t), b)$ the PDE that we derived before holds