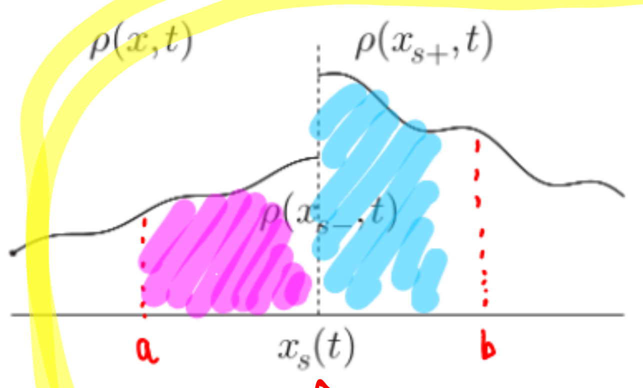
[14-May-2024] Final Exam

The final exam will be given Wednesday, May 14 from 12:45pm-2:45 in AB634. Here is a sample final to help you prepare for the exam.

$$\rho(x_s(t)^-,t)\,x_s'(t) + \int_a^{x_s(t)} \frac{\partial \rho}{\partial t}(x,t)\,dx - \rho(x_s(t)^+,t)\,x_s'(t) + \int_{x_s(t)}^{b} \frac{\partial \rho}{\partial t}(x,t)\,dx = q(a,t) - q(b,t)$$

- Solve for $x_s'(t)$ to see how the shock moves on the separate intervals $(a, x_s(t))$ and $(x_s(t), b)$ the PDE that we derived before holds

$$x_s'(t)\big(\rho(x_s(t)^-,t) - \rho(x_s(t)^+,t)\big) = q(a,t) - q(b,t) - \int_a^{x_s(t)} \frac{\partial \rho}{\partial t}\,dx - \int_{x_s(t)}^{b} \frac{\partial \rho}{\partial t}\,dx$$



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tempting to join these integrals back together but the derivative doesn't exist at $x_s(t)$

differential equation does not hold at $x_s(t)$ only on the left and right parts.

PDE

$$\frac{\partial}{\partial t}\rho(x,t) + \boxed{\frac{\partial}{\partial x} q(x,t)} = 0 .$$

with respect to x

$$\int_a^{x_s(t)} \frac{\partial \rho}{\partial t}\,dx = \int_a^{x_s(t)} -\frac{\partial q}{\partial x}\,dx = -\big(q(x_s(t)^-,t) - q(a,t)\big)$$

$$\int_{x_s(t)}^{b} \frac{\partial \rho}{\partial t}\,dx = \int_{x_s(t)}^{b} -\frac{\partial q}{\partial x}\,dx = -\left(q(b,t) - q(x_s(t)^+, t)\right)$$

$$q(a,t) - q(b,t) - \int_a^{x_s(t)} \frac{\partial \rho}{\partial t}\,dx - \int_{x_s(t)}^{b} \frac{\partial \rho}{\partial t}\,dx$$

$$x_s'(t)\left(\rho(x_s(t)^-, t) - \rho(x_s(t)^+, t)\right)$$

$$= \cancel{q(a,t)} - \cancel{q(b,t)} + q(x_s(t)^-, t) - \cancel{q(a,t)} + \cancel{q(b,t)} - q(x_s(t)^+, t)$$

Therefore

$$x_s'(t) = \frac{q(x_s(t)^-, t) - q(x_s(t)^+, t)}{\rho(x_s(t)^-, t) - \rho(x_s(t)^+, t)}$$

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$$\boxed{\frac{dx_s}{dt} = \frac{q(x_{s+}, t) - q(x_{s-}, t)}{\rho(x_{s+}, t) - \rho(x_{s-}, t)} = \frac{[q]}{[\rho]},} \qquad (12.6.20)$$

Example

$$1\,\frac{\partial \rho}{\partial t} + 2\rho \frac{\partial \rho}{\partial x} = 0$$

$$\rho(x, 0) = \begin{cases} 4 & x < 0 \\ 3 & x > 0. \end{cases}$$

$t_0 = 0$

already a shock in the initial distribution of cars..

Solve this by characteristics for practice

Since the coefficient in front of $\frac{\partial \rho}{\partial t}$ is 1 and $t_0 = 0$ then let's use t to parameterize the characteristics rather than introducing a new variable s and then solving for $s = t$.

So let $x = x(t)$ then

$$\frac{d}{dt}\rho(x(t), t) = \frac{\partial \rho}{\partial x} x'(t) + \frac{\partial \rho}{\partial t}$$

So if $x'(t) = 2\rho$ then $\frac{d\rho}{dt} = 0$

On the characterist $\frac{d\rho}{dt} = 0$ implies $\rho(x(t), t) = c_1$

find the const with this

$$\rho(x,0) = \begin{cases} 4 & x \le 0 \\ 3 & x > 0. \end{cases}$$

$$\rho(\underbrace{x(0)}_{x_0}, 0) = \begin{cases} 4 & \text{if } x(0) \le 0 \\ 3 & \text{if } \underbrace{x(0)}_{x_0} > 0 \end{cases}$$

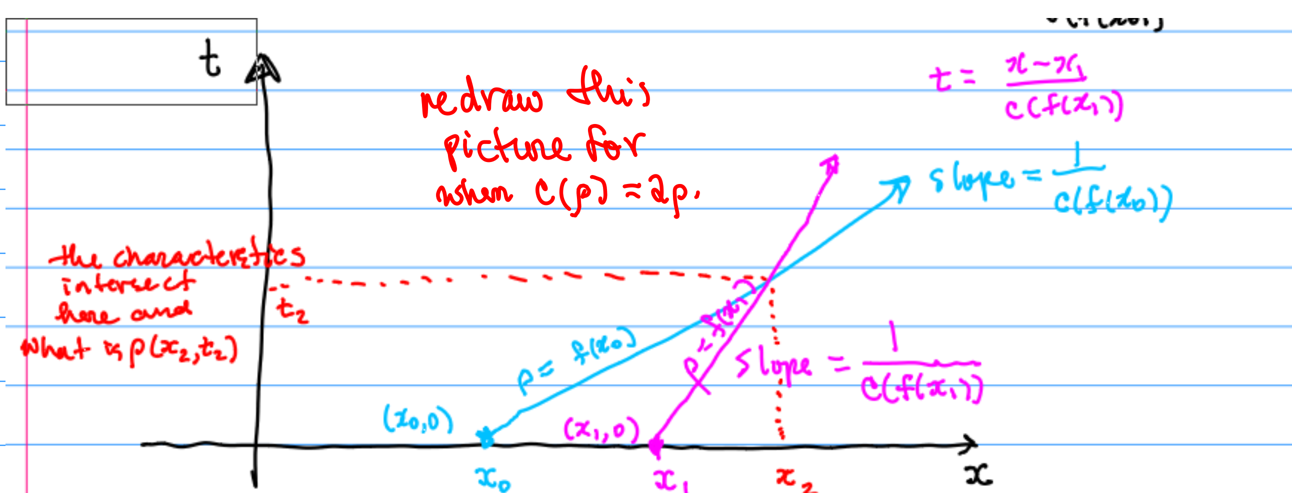
Thus $x'(t) = 2\rho = 2c_1$

$\therefore$ $x(t) = 2c_1 t + c_2$

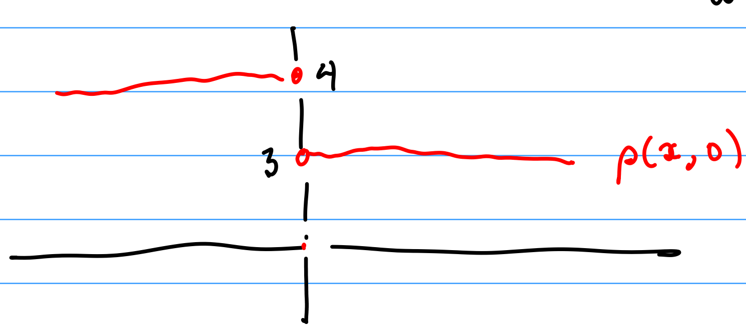
$x(0) = 2c_1 \cdot 0 + c_2$ thus $c_2 = \underbrace{x(0)}_{x_0}$

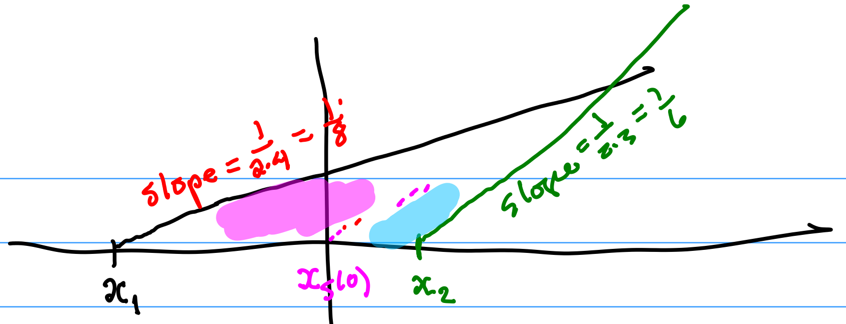
Thus $\rho(2c_1 t + x_0, t) = c_1$ where $c_1 = \rho(x_0, 0)$

want to solve for $\rho(x,t)$



$x(t) = 2c_1 t + x_0$ $\qquad t = \frac{x - x_0}{2c_1}$





how does $x_s(t)$ evolve?

$$x_s'(t) = \frac{q(x_s(t)^-, t) - q(x_s(t)^+, t)}{\rho(x_s(t)^-, t) - \rho(x_s(t)^+, t)}$$

to find q in terms of ρ compare the general PDE to the one in the example

$$\frac{\partial \rho}{\partial t} + 2\rho \frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial}{\partial t} \rho(x,t) + \frac{\partial}{\partial x} q(x,t) = 0 .$$

with respect

Therefore

$$\frac{\partial q}{\partial x} = 2\rho \frac{\partial \rho}{\partial x} = \frac{\partial \rho^2}{\partial x}$$ thus $\boxed{q = \rho^2 + c_3}$

it doesn't matter what c_3 is because it cancels here

$$x_s'(t) = \frac{\rho(x_s(t)^-, t)^2 - \rho(x_s(t)^+, t)^2}{\rho(x_s(t)^-, t) - \rho(x_s(t)^+, t)} = \frac{16 - 9}{4 - 3} = 7$$

$$x_s(t) = 7t + c_4$$

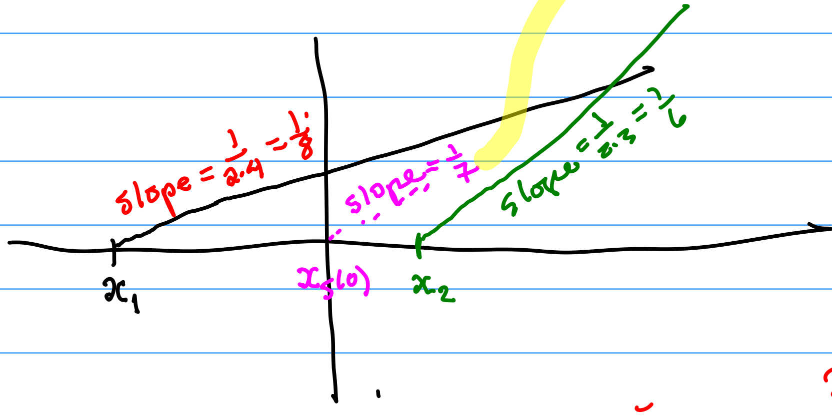
$x_s(0) = 0$

initial location of discontinuity

so $c_4 = 0$

$$\boxed{x_s(t) = 7t}$$

$$\rho(x,t) = \begin{cases} 4 & \text{for } x \le x_s(t) \\ 3 & \text{for } x > x_s(t) \end{cases} = \begin{cases} 4 & \text{for } x \le 7t \\ 3 & \text{for } x > 7t \end{cases}$$



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