1. [Kincaid and Cheney Problem 6.8#6] Show that the Hilbert matrix with elements $a_{ij} = (i + j + 1)^{-1}$ for $i, j = 0, 1, 2, \ldots, n - 1$ is a Gram matrix for the functions $1, x, x^2, \ldots, x^{n-1}$.

We define the inner product

$$\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx$$

on the space of functions $L^2\big([0,1]; \mathbf{R}\big)$ and note that

$$1, x, x^2, \ldots, x^{n-1} \in L^2\big([0,1]; \mathbf{R}\big).$$

The corresponding Gram matrix has elements

$$\langle x^i, x^j \rangle = \int_0^1 x^{i+j}\, dx = \frac{1}{i+j+1} x^{i+j+1}\Big|_0^1 = \frac{1}{i+j+1}$$

which are the entries of the Hilbert matrix.

2. [Kincaid and Cheney Problem 6.8#8] In the three-term recurrence relation for the orthogonal polynomials, assume that the inner product is

$$\langle f, g\rangle = \int_{-a}^{a} f(x)g(x)w(x)dx$$

where w is an even function. Prove that $a_n = 0$ for all n. Prove that p_n is even if n is even and that p_n is odd if n is odd.

By definition $p_0 = 1$ which is even. Moreover

$$a_1 = \frac{\langle xp_0, p_0\rangle}{\langle p_0, p_0\rangle} = \frac{\int_{-a}^{a} x\,dx}{\int_{-a}^{a} 1\,dx} = 0$$

and consequently $p_1 = x - a_1 = x$ is odd. It further follows that

$$a_2 = \frac{\langle xp_1, p_1\rangle}{\langle p_1, p_1\rangle} = \frac{\int_{-a}^{a} x^3\,dx}{\int_{-a}^{a} x^2\,dx} = 0.$$

We now procede by induction which can be stated as follows:

> Suppose p_k is even if k is even and that p_k is odd if k is odd for all $k < n$, then p_n is even if n is even and p_n is odd if n is odd.

Notice no matter whether k is even or odd that the function $x\big(p_k(x)\big)^2$ is odd. This can be seen by the following two equalities:

Case k is even:

$$(-x)\big(p_k(-x)\big)^2 = -x\big(p_k(x)\big)^2$$

Case k is odd:

$$(-x)\big(p_k(-x)\big)^2 = -x\big(-p_k(x)\big)^2 = -x\big(p_k(x)\big)^2$$

Therefore

$$a_n = \frac{\langle xp_{n-1}, p_{n-1}\rangle}{\langle p_{n-1}, p_{n-1}\rangle} = \frac{\int_{-a}^{a} x\big(p_{n-1}(x)\big)^2\,dx}{\int_{-a}^{a} \big(p_{n-1}(x)\big)^2\,dx} = 0$$

and consequently

$$p_n(x) = (x - a_n)p_{n-1}(x) - b_n p_{n-2}(x) x p_{n-1}(x) - b_n p_{n-2}(x).$$

We now consider the case when n is even and n odd separately.

Case n is odd: By the induction hypothesis p_{n-1} is even and p_{n-1} is odd. It follows that $xp_{n-1}(x)$ is odd and therefore $p_n(x)$, being the sum of two odd functions, is again odd.

Case n is even: By the induction hypothesis p_{n-1} is odd and p_{n-1} is even. It follows that $xp_{n-1}(x)$ is even and therefore $p_n(x)$, being the sum of two even functions, is again even.

This completes the induction and the proof.

3. [Kincaid and Cheney Problem 6.8#21] Derive these Legendre polynomials:

$$
\begin{aligned}
p_3(x) &= x^3 - \tfrac{3}{5}x \\
p_4(x) &= x^4 - \tfrac{6}{7}x^2 + \tfrac{3}{35} \\
p_5(x) &= x^5 - \tfrac{10}{9}x^3 + \tfrac{5}{21}x
\end{aligned}
$$

I wrote a Maple script to implement the calculation given as Theorem 5 in Kincaid and Cheney on page 400. The script is

```
# Kincaid and Cheney Problem 6.8 # 21
# Written December 4 by Eric Olson for Math 761
restart;
kernel(printbytes=false):
l2prod:=(f,g)->int(f*g,x=-1..1);
p[0]:=1;
a[1]:=l2prod(x*p[0],p[0])/l2prod(p[0],p[0]);
p[1]:=x-a[1];
for n from 2 to 5 do
    a[n]:=l2prod(x*p[n-1],p[n-1])/l2prod(p[n-1],p[n-1]);
    b[n]:=l2prod(x*p[n-1],p[n-2])/l2prod(p[n-2],p[n-2]);
    p[n]:=sort(collect((x-a[n])*p[n-1]-b[n]*p[n-2],x));
od;
```

and the output is

```
    |\^/|     Maple 9.5 (IBM INTEL LINUX)
._|\|   |/|_. Copyright (c) Maplesoft, a division of Waterloo Maple Inc. 2004
 \  MAPLE  /  All rights reserved. Maple is a trademark of
 <____ ____>  Waterloo Maple Inc.
      |       Type ? for help.
# Kincaid and Cheney Problem 6.8 # 21
# Written December 4 by Eric Olson for Math 761
> restart;
> kernel(printbytes=false):
> l2prod:=(f,g)->int(f*g,x=-1..1);
                                          1
                                         /
                                        |
                   l2prod := (f, g) ->  |   f g dx
                                        |
                                       /
                                        -1


> p[0]:=1;
                              p[0] := 1

> a[1]:=l2prod(x*p[0],p[0])/l2prod(p[0],p[0]);
                              a[1] := 0

> p[1]:=x-a[1];
                              p[1] := x

> for n from 2 to 5 do
```

```
>     a[n]:=l2prod(x*p[n-1],p[n-1])/l2prod(p[n-1],p[n-1]);
>     b[n]:=l2prod(x*p[n-1],p[n-2])/l2prod(p[n-2],p[n-2]);
>     p[n]:=sort(collect((x-a[n])*p[n-1]-b[n]*p[n-2],x));
> od;
                                  a[2] := 0

                                 b[2] := 1/3

                                      2
                               p[2] := x  - 1/3

                                  a[3] := 0

                                 b[3] := 4/15

                                     3
                              p[3] := x  - 3/5 x

                                  a[4] := 0

                                 b[4] := 9/35

                                 4        2
                          p[4] := x  - 6/7 x  + 3/35

                                  a[5] := 0

                                         16
                                  b[5] := --
                                         63

                               5          3
                        p[5] := x  - 10/9 x  + 5/21 x

> quit
bytes used=753184, alloc=655240, time=0.10
```

4. [Kincaid and Cheney Problem 6.9#2] Find the best approximation of $\sqrt{x}$ by a first-degree polynomial on the interval $[0,1]$.

In light of Corollary 2 in Kincaid and Cheney page 408 and Example 1 on the preceding page we solve the following system of equations:

$$\begin{aligned} g(0) - f(0) &= \delta \\ g(\xi) - f(\xi) &= -\delta \\ g(1) - f(1) &= \delta \\ g'(\xi) - f'(\xi) &= 0 \end{aligned}$$

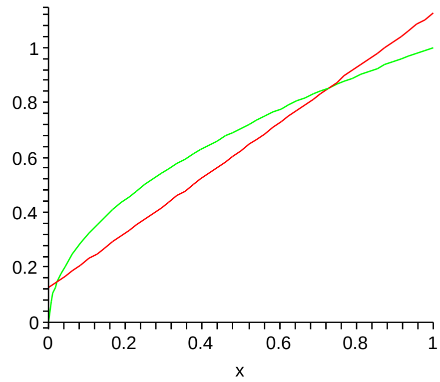
where $g(x) = ax + b$ and $f(x) = \sqrt{x}$. The Maple script

```
# Kincaid and Cheney Problem 6.9 # 2
# Written December 4 by Eric Olson for Math 761
restart;
kernel(printbytes=false):
eq1:=g(0)-f(0)=delta;
eq2:=g(xi)-f(xi)=-delta;
eq3:=g(1)-f(1)=delta;
eq4:=D(g)(xi)-D(f)(xi)=0;
g:=x->a*x+b;
f:=sqrt;
eqns:={eq1,eq2,eq3,eq4};
S:=solve(eqns,{a,b,xi,delta});
g1:=subs(S,g(x));
```

solves these equations. The best approximation is

$$f(x) = x + 1/8$$

which has a graph



The Maple output follows:

```
    |\^/|     Maple 9.5 (IBM INTEL LINUX)
._|\|   |/|_. Copyright (c) Maplesoft, a division of Waterloo Maple Inc. 2004
 \  MAPLE  /  All rights reserved. Maple is a trademark of
 <____ ____>  Waterloo Maple Inc.
      |       Type ? for help.
```

```
# Kincaid and Cheney Problem 6.9 # 2
# Written December 4 by Eric Olson for Math 761
> restart;
> kernel(printbytes=false):
> eq1:=g(0)-f(0)=delta;
                          eq1 := g(0) - f(0) = delta

> eq2:=g(xi)-f(xi)=-delta;
                        eq2 := g(xi) - f(xi) = -delta

> eq3:=g(1)-f(1)=delta;
                          eq3 := g(1) - f(1) = delta

> eq4:=D(g)(xi)-D(f)(xi)=0;
                        eq4 := D(g)(xi) - D(f)(xi) = 0

> g:=x->a*x+b;
                              g := x -> a x + b

> f:=sqrt;
                                  f := sqrt

> eqns:={eq1,eq2,eq3,eq4};
eqns :=

                             1/2                                     1
    {b = delta, a xi + b - xi    = -delta, a + b - 1 = delta, a - ------- = 0}
                                                                       1/2
                                                                   2 xi

> S:=solve(eqns,{a,b,xi,delta});
                  S := {b = 1/8, a = 1, xi = 1/4, delta = 1/8}

> g1:=subs(S,g(x));
                                g1 := x + 1/8

> quit
bytes used=1419008, alloc=1179432, time=0.14
```

5. [Kincaid and Cheney Problem 6.9#3] Show that the subspaces in $C[0,1]$ spanned by these sets are Haar subspaces:

$$A = \{1, x^2, x^3\}, \qquad B = \{1, e^x, e^{2x}\}, \qquad C = \{(x+2)^{-1}, (x+3)^{-1}, (x+4)^{-1}\}.$$

Let $a + bx^2 + cx^3$ be in the span of A. Claim that this element has at most two roots in the interval $[-1, 1]$. if $c = 0$ then $a + bx^2$ clearly has at most two roots. If $c \neq 1$ we may consider the polynmial $\alpha + \beta x + x^3$ were $\alpha = a/c$ and $\beta = b/c$. Suppose for contradiction there were three distinct roots $0 \leq x_1 < x_2 < x_3 \leq 1$ such that

$$\begin{aligned} \alpha + \beta x^2 + x^3 &= (x - x_1)(x - x_2)(x - x_3) \\ &= x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_1x_3 + x_2x_3)x - x_1x_2x_3. \end{aligned}$$

Equating coefficients we obtain $0 = x_1x_2 + x_1x_3 + x_2x_3 \geq x_2x_3 > 0$ which is a contradiction.

Let $a + be^x + ce^{2x}$ be in the span of B. Writing $w = e^x$ we may consider the polynomial $a + bw + cw^2$ which has at most 2 roots w_1 and w_2. Since the function $x \to e^x$ is injective on $[0, 1]$ there are at most two numbers x_1 and x_2 such that $w_1 = e^{x_1}$ and $w_2 = e^{x_2}$. It follows that $a + be^x + ce^{2x} = 0$ has at most 2 solutions.

Let $a(x+2)^{-1} + b(x+3)^{-1} + c(x+4)^{-1}$ be in the span of C. Finding a common denominator shows this expression is equal to

$$\frac{a(x+3)(x+4) + b(x+2)(x+4) + c(x+2)(x+3)}{(x+1)(x+2)(x+3)} = \frac{p(x)}{(x+1)(x+2)(x+3)}$$

for some polynomial $p(x)$ of degree less than or equal 2. Since $p(x)$ has at most 2 roots than $a(x+2)^{-1} + b(x+3)^{-1} + c(x+4)^{-1} = 0$ has at most 2 solutions.

6. [Kincaid and Cheney Problem 6.9#4] Show that the subspaces in $C[-1,1]$ spanned by these sets are Haar subspaces:

$$A = \{1, x^2, x^3\}, \qquad B = \{|x|, |x-1|\}, \qquad C = \{e^x, x+1\}.$$

We consider again the equation $x_1x_2 + x_1x_3 + x_2x_3 = 0$ from part A of the previous problem and this equation has solutions such that $-1 \le x_1 < x_2 < x_3 \le 1$. In particular, if $x_2 = 1/2$ and $x_3 = 1/3$, then

$$x_1 = \frac{-x_2x_3}{x_2 + x_3} = -\frac{1}{5} \in [-1,1].$$

Therefore the span of A is not a Haar subspace.

Consider the function $f(x) = 2|x| - |x-1|$ in the span of B. Clearly $f(1) = 2 - 1 = 0$ and $f(1/3) = 2/3 - |-2/3| = 0$ therefore B is not a Haar subspace.

Consider the function $f(x) = 1 + x - \frac{4}{5}e^x$. Since

$$\begin{aligned} f(-1) &= -\frac{4}{5}e^{-1} < 0 \\ f(0) &= 1 - \frac{4}{5} > 0 \\ f(1) = 2 - \frac{4}{5}e &< 4 - \frac{4}{5}(2.7) < 0, \end{aligned}$$

then by the intermediate value property of continuous functions there must be points x_1 and x_2 such that

$$-1 < x_1 < 0 < x_2 < 1 \qquad \text{and} \qquad f(x_1) = f(x_2) = 0.$$

Therefore C is not a Haar subspace.

7. [Kincaid and Cheney Problem 6.9#8] Prove the quadratic polynomial of best approximation to the function $\cosh x$ on the interval $[-1, 1]$ is $a + bx^2$ where $b = \cosh 1 - 1$ and a is obtained by simultaneously solving for a and t in the system

$$\begin{cases} 2a = 1 + \cosh t - t^2 b \\ \sinh t = 2tb. \end{cases}$$

Define $f(x) = \cosh x$. First we claim the quadratic polynomial that best approximates f on the interval $[-1, 1]$ is of the form $a+bx^2$. Let $F = \{a+bx^2 : a, b \in \mathbf{R}\}$, $G = \{cx : x \in R\}$ and $P_2 = \{a + cx + bx^2 : a, b, c \in \mathbf{R}\}$. Let a, b and c be chosen so that

$$\big\|f - (a + cx^2 + bx^2)\big\| = \min\big\{\|f - g\| : g \in P_2\big\}.$$

Define $f_2(x) = \cosh x-(a+bx^2)$. Since $f_2(x)$ is even then $x \in \operatorname{crit}(f_2)$ implies $-x \in \operatorname{crit}(f_2)$ and moreover $f_2(x)$ and $f_2(-x)$ have the same sign. Thus, there is no function in G that has the same signs as f_2 on $\operatorname{crit}(f_2)$. It follows that Kolmogorov's Characterization Theorem from Kincaid and Cheney page 407 implies $\|f_2\| = \operatorname{dist}(f_2, G)$. In particular

$$\big\|f_2 - (a + cx^2 + bx^2)\big\| = \operatorname{dist}(f_2, G) = \|f_2\|$$

and therefore there is a function of the form $a + bx^2$ such that

$$\big\|f - (a + bx^2)\big\| = \min\big\{\|f - g\| : g \in P_2\big\}.$$

Now make the change of variables $y = \sqrt{x}$. To obtain the following equivalent minimization problem: Find $a + by$ that best approximates the function $\cosh\sqrt{y}$ on the interval $[0, 1]$.

This problem is in the form covered by Corollary 2 from Kincaid and Cheney page 408 so we may obtain the solution by solving the system of equations:

$$\begin{aligned} g_3(0) - f_3(0) &= \delta \\ g_3(\xi) - f_3(\xi) &= -\delta \\ g_3(1) - f_3(1) &= \delta \\ g_3'(\xi) - f_3'(\xi) &= 0 \end{aligned}$$

where $g_3(y) = a + by$ and $f_3(y) = \cosh\sqrt{y}$. Simplifying obtains

$$\begin{aligned} a - 1 &= \delta \\ a + b\xi - \cosh\sqrt{\xi} &= -\delta \\ a + b - \cosh 1 &= \delta \\ 2b\sqrt{\xi} - \sinh\sqrt{\xi} &= 0 \end{aligned}$$

Elimination of δ from the 2nd and 3rd equation, setting $t = \sqrt{\xi}$ and futher simplification obtains the desired result

$$\begin{aligned} 2a &= 1 + \cosh t - t^2 b \\ b &= \cosh 1 - 1 \\ \sinh t &= 2bt. \end{aligned}$$