Math 701 Quiz 1 Version A

INSTRUCTIONS: Complete 3 questions out of the 6 questions below. Clearly indicate which problems you wish graded. Work each problem on a separate sheet of paper (or more if necessary).

1. For $x \in \mathbf{R}^n$ and $A \in \mathbf{R}^{n\times n}$ define

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$
$$\|A\|_2 = \max\{\, \|Ax\|_2 : \|x\| = 1 \,\}$$

and

$$\rho(A) = \max\{\, |\lambda| : \ \lambda \text{ is an eigenvalue of } A \,\}.$$

Prove that $\|A\|_2 = \rho(A^T A)^{1/2}$.

2. Prove that every nonconstant polynomial has at least one root in the complex field.

3. Let $A \in \mathbf{R}^{n\times n}$ be a symmetric positive definite matrix and $\{u^{(1)}, u^{(2)}, \ldots, u^{(n)}\}$ be a collection of vectors in $\mathbf{R}^n$ such that

$$u^{(i)} \cdot Au^{(j)} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Let $x^{(0)} \in \mathbf{R}^n$ be arbitrary and define

$$x^{(i)} = x^{(i-1)} + \langle b - Ax^{(i-1)}, u^{(i)} \rangle u^{(i)} \qquad \text{for} \qquad i = 1, 2, \ldots, n.$$

Prove that $Ax^{(n)} = b$.

4. Suppose $A \in \mathbf{R}^{n\times n}$ is symmetric and positive definite. Prove that the problem of solving $Ax = b$ is equivalent to the problem of minimizing the quadratic form

$$q(x) = \langle x, Ax \rangle - 2\langle x, b \rangle.$$

5. Let f'' be continuous and let r be a simple zero of f. Prove there is a neighborhood of r and a constant C such that if Newton's method is started in that neighborhood, the successive points become steadily closer to r and satisfy

$$|x_{n+1} - r| \le C(x_n - r)^2 \qquad \text{for} \qquad n = 0, 1, 2, 3, \ldots.$$

6. Let $A \in \mathbf{R}^{n\times n}$ and $b, x \in \mathbf{R}^n$ such that $Ax = b$. Given $\tilde{x} \in \mathbf{R}^n$ prove that

$$\frac{1}{\kappa(A)} \frac{\|b - A\tilde{x}\|}{\|b\|} \le \frac{\|x - \tilde{x}\|}{\|x\|} \le \kappa(A) \frac{\|b - A\tilde{x}\|}{\|b\|}$$

where $\kappa(A) = \|A\|\|A^{-1}\|$.