Math 701 Programming Project 2

Iterated Solutions to Systems of Linear Equations

We will work on the programming part of this project in class; however, your final report should be prepared independently. Where appropriate include full program listings and output.

1. Let $g\colon \mathbf{R}^d \to \mathbf{R}^d$ be a differentiable function such that $\|g'(x)\| < 1$ for all $x \in \mathbf{R}^d$. Prove or disprove the claim that the iteration $x_{n+1} = g(x_n)$ converges for any initial vector $x_0 \in \mathbf{R}^d$.

2. Given a matrix $A \in \mathbf{R}^{d\times d}$ and a vector $b \in \mathbf{R}^d$, suppose there exists an invertible matrix $R \in \mathbf{R}^{d\times d}$ such that $\|I - R^{-1}A\| < 1$. Define $g\colon \mathbf{R}^d \to \mathbf{R}^d$ by

$$g(x) = R^{-1}\big(b + (R - A)x\big).$$

(i) Prove the iteration $x_{n+1} = g(x_n)$ converges to a fixed point $p \in \mathbf{R}^d$ such that $g(p) = p$ for any initial vector $x_0 \in \mathbf{R}^d$.

(ii) Show $g(p) = p$ implies $Ap = b$.

(iii) Is it true or false that A is invertible if and only if there exists $R \in \mathbf{R}^{d\times d}$ such that $\|I - R^{-1}A\| < 1$. Explain your reasoning.

3. Suppose the matrix A is strictly diagonally dominant with entries a_{ij} such that

$$|a_{ii}| > 0 \quad \text{and} \quad |a_{ii}| > \sum_{j\neq i} |a_{ij}| \qquad \text{for every} \qquad i = 1, \ldots, d.$$

Let R be the diagonal matrix with a_{ii} on its diagonal. Show that $\|I - R^{-1}A\|_\infty < 1$. Note that the method of solving systems of linear equations by taking R to be the diagonal part of A is called Jacobi iteration.

4. Consider the matrix A and vector b given by

$$A = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Let $R \in \mathbf{R}^{4\times 4}$ be the diagonal matrix with 3's on it's diagonal. Find $\|I - R^{-1}A\|_\infty$ and write a program that uses the iteration $x_{n+1} = g(x_n)$ with starting value $x_0 = b$ to solve the equation $Ax = b$. Print the vectors x_n for $n = 0, \ldots 10$.

5. Consider the matrix A and vector b given by

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Note that A is not strictly diagonally dominant. It does, however, satisfy a weaker condition and is simply called diagonally dominant. Let $R \in \mathbf{R}^{4\times 4}$ be the diagonal matrix with 2's on it's diagonal.

(i) Show that $\|I - R^{-1}A\|_\infty = 1$ and $\|I - R^{-1}A\|_1 = 1$.

(ii) Write a program to compute the spectral norm $\|I - R^{-1}A\|_2$ using the power method. It is fine to modify the program written in class.

(iii) Is $\|I - R^{-1}A\|_2$ strictly less than one?

(iv) Write a program that uses the iteration $x_{n+1} = g(x_n)$ with starting value $x_0 = b$ to solve the equation $Ax = b$. Print the vectors x_n for $n = 0, \ldots 10$.

6. [Extra Credit] Suppose the matrix A has entries a_{ij} such that

$$|a_{ii}| > 0 \quad \text{and} \quad |a_{ii}| \geq \sum_{j\neq i} |a_{ij}| \qquad \text{for every} \qquad i = 1, \ldots, d$$

where the second inequality is strict for at least one value of i. Let R be the diagonal matrix with a_{ii} on its diagonal. Prove or disprove that $\|I - R^{-1}A\|_2 < 1$.