Math 701 Quiz 2 Version A

1. Find a suitable trigonometric identity so that \(1 - \cos x\) can be accurately computed for small \(x\) with calls to the system functions for \(\sin x\) or \(\cos x\).

Recall the sine angle addition formula

\[
\sin(a + b) = \sin a \cos b + \cos a \sin b.
\]

This formula is easy to remember because if its symmetry. Differentiate with respect to \(a\) to obtain the cosine angle addition formula

\[
\cos(a + b) = \cos a \cos b - \sin a \sin b.
\]

Now, setting \(a = x/2\) and \(b = x/2\) yields the half-angle formula

\[
\cos x = \cos^2(x/2) - \sin^2(x/2) .
\]

Subtracting the above identity from the Pythagorean theorem \(1 = \cos^2(x/2) + \sin^2(x/2)\) results in the trigonometric identity \(1 - \cos x = 2\sin^2(x/2)\) which is suitable to accurately approximate \(1 - \cos x\) for small values.

2. State Taylor’s theorem including all hypothesis and the remainder term.

**Taylor’s Theorem.** Let \(f: \mathbb{R} \to \mathbb{R}\) be an \(n+1\) times continuously differentiable function. Then

\[
f(x + h) = \sum_{k=0}^{n} \frac{h^k}{n!} f^{(n)}(x) + \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(\xi)
\]

for some \(\xi\) between \(x\) and \(x + h\).

3. State the power method for finding the largest eigenvalue and corresponding eigenvector of a symmetric positive semidefinite matrix \(A \in \mathbb{R}^{d \times d}\).

Let \(x_0 \in \mathbb{R}^d\) be chosen randomly and define

\[
y_n = Ax_n \quad \text{and} \quad x_{n+1} = y_n / \|y_n\| \quad \text{for} \quad n = 0, 1, 2, \ldots .
\]

Then

\[
\|y_n\| \to \lambda \quad \text{and} \quad x_n \to \xi \quad \text{as} \quad n \to \infty
\]

where \(\lambda\) is the largest eigenvalue of \(A\) and \(\xi\) is its corresponding eigenvector.
4. Let $f$ be a twice continuously differentiable function and $p$ be a point such that $f(p) = 0$ and $f'(p) \neq 0$. Prove that Newton’s method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ is quadratically convergent provided $x_0$ is close enough to $p$.

Since $f'(p) \neq 0$ there exists $\delta > 0$ such that
\[ A = \min\{ |f'(t)| : t \in [p - \delta, p + \delta] \} > 0. \]

Further define
\[ B = \max\{ |f''(t)| : t \in [p - \delta, p + \delta] \}. \]

Now let $\epsilon = \min\{ \delta, M^{-1} \}$ where $M = B/(2A)$. We claim the condition $|x_0 - p| < \epsilon$ is sufficient to guarantee
\[ \lim_{n \to \infty} x_n = p \quad \text{and} \quad |x_{n+1} - p| \leq M|x_n - p|^2 \quad \text{for} \quad n = 0, 1, 2, \ldots. \]

Define $e_n = x_n - p$. By Taylor’s theorem there is $\xi_n$ between $p$ and $x_n$ such that
\[ 0 = f(p) = f(x_n) + (p - x_n)f'(x_n) + \frac{1}{2}(p - x_n)^2 f''(\xi_n). \]

It follows that
\[ 0 = \frac{f(x_n)}{f'(x_n)} - e_n + \frac{1}{2} e_n^2 \frac{f''(\xi_n)}{f'(x_n)} \quad \text{or equivalently} \quad e_n - \frac{f(x_n)}{f'(x_n)} = \frac{1}{2} e_n^2 \frac{f''(\xi_n)}{f'(x_n)}. \]

Consequently
\[ |e_{n+1}| = |x_{n+1} - p| = \left| x_n - \frac{f(x_n)}{f'(x_n)} - p \right| = \left| e_n - \frac{f(x_n)}{f'(x_n)} \right| = \frac{1}{2} |e_n|^2 \left| \frac{f''(\xi_n)}{f'(x_n)} \right|. \]

Suppose for induction that $|x_n - p| < \epsilon$ as is the case when $n = 0$. Then $\epsilon \leq \delta$ implies
\[ |e_{n+1}| = \frac{1}{2} |e_n|^2 \left| \frac{f''(\xi_n)}{f'(x_n)} \right| \leq \frac{1}{2} \max\{ |f''(t)| : t \in [p - \epsilon, p + \epsilon] \} \leq M|e_n|^2. \]

Since $M|e_n| \leq M\epsilon \leq 1$ then $|e_{n+1}| \leq |e_n|$ which implies $|x_{n+1} - p| < \epsilon$ and completes the induction. In particular, we have shown that
\[ |e_{n+1}| \leq M|e_n|^2 \quad \text{and} \quad |e_{n+1}| \leq |e_n| \quad \text{for} \quad n = 0, 1, 2, \ldots. \]

It remains to show $x_n \to p$ as $n \to \infty$. The second inequality above immediately implies $|e_n| \leq |e_0|$. Define $\gamma = M|e_0|$. Since $M|e_0| < M\epsilon \leq MM^{-1} = 1$ then $\gamma < 1$. Now
\[ |e_{n+1}| \leq M|e_n|^2 \leq (M|e_0|)|e_n| = \gamma |e_n| \]

implies $|e_n| \leq \gamma^n |e_0|$ for all $n$. Since $\gamma^n \to 0$ as $n \to \infty$ it follows that $x_n \to p$. 
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5. Let $B \in \mathbb{R}^{d \times d}$ and consider the matrix norm given by

$$
\|B\|_2 = \max\{ \|Bx\|_2 : \|x\|_2 = 1 \}
$$

where $\|x\|_2 = \left( \sum_{i=1}^{d} |x_i|^2 \right)^{1/2}$.

Prove $\|B\|_2 = \rho(B^T B)^{1/2}$ where $\rho(A) = \max\{ |\lambda| : \lambda$ is an eigenvalue of $A \}$.

Let $A = B^T B$. Then $A \in \mathbb{R}^{d \times d}$ is a symmetric positive semidefinite matrix. The spectral theorem for real symmetric matrices implies that there exists an orthonormal basis of eigenvectors $\xi_i$ with corresponding eigenvalues $\lambda_i$ for $i = 1, 2, \ldots, d$ such that

$$
A \xi_i = \lambda_i \xi_i \quad \text{and} \quad \xi_i : \xi_j = \begin{cases} 
1 & \text{for } i = j \\
0 & \text{otherwise}
\end{cases}
$$

Since $A$ is semidefinite it is further the case that $\lambda_i \geq 0$ for all $i$. Given $x \in \mathbb{R}^d$ there exists constants $c_i \in \mathbb{R}$ such that

$$
x = \sum_{k=1}^{d} c_k \xi_k.
$$

Consequently, the orthonormality of the $\xi_k$’s implies

$$
\|x\|_2^2 = x : x = \sum_{k=1}^{d} c_k \xi_k \cdot \sum_{\ell=1}^{d} c_\ell \xi_\ell = \sum_{k=1}^{d} \sum_{\ell=1}^{d} c_k c_\ell \xi_k : \xi_\ell = \sum_{k=1}^{d} c_k^2.
$$

Similarly

$$
\|Bx\|_2^2 = Bx : Bx = x : Ax = \sum_{k=1}^{d} c_k \xi_k \cdot \sum_{\ell=1}^{d} c_\ell \lambda_\ell \xi_\ell = \sum_{k=1}^{d} \lambda_k c_k^2.
$$

Since $\lambda_k \geq 0$ then $\lambda_k = |\lambda_k|$ and it follows that

$$
\|B\|_2 = \max\left\{ \sum_{k=1}^{d} \lambda_k c_k^2 : \sum_{k=1}^{d} c_k^2 = 1 \right\} = \max\left\{ \sum_{k=1}^{d} |\lambda_k| c_k^2 : \sum_{k=1}^{d} c_k^2 = 1 \right\}.
$$

The above maximum may be interpreted as the maximum over all possible weighted averages of the $|\lambda_k|$’s. Since any weighted average is between the smallest and largest, then further placing all the weight on the largest $|\lambda_k|$ yields that

$$
\|B\|_2^2 = \max\{ |\lambda_k| : k = 1, 2, \ldots, d \}
$$

or equivalently that $\|B\|_2 = \rho(B^T B)^{1/2}$. 
6. Consider the matrix \( A \in \mathbb{R}^{4 \times 4} \) given by
\[
A = \begin{bmatrix}
-8 & -6 & -6 & -1 \\
4 & 8 & -4 & 1 \\
-4 & -10 & 2 & -8 \\
3 & -7 & 9 & 7
\end{bmatrix}
\]

(i) Find \( \|A\|_1 \).

Since
\[
\sum_{j=1}^{d} |a_{1,j}| = 8 + 4 + 4 + 3 = 19 \\
\sum_{j=1}^{d} |a_{2,j}| = 6 + 8 + 10 + 7 = 31 \\
\sum_{j=1}^{d} |a_{3,j}| = 6 + 4 + 2 + 9 = 21 \\
\sum_{j=1}^{d} |a_{4,j}| = 1 + 1 + 8 + 7 = 17
\]

then
\[
\|A\|_1 = \max \left\{ \sum_{j=1}^{d} |a_{ij}| : i = 1, \ldots, d \right\} = \max\{19, 31, 21, 17\} = 31.
\]

(ii) Find \( \|A\|_\infty \).

Since
\[
\sum_{i=1}^{d} |a_{i,1}| = 8 + 6 + 6 + 1 = 21 \\
\sum_{i=1}^{d} |a_{i,2}| = 4 + 8 + 4 + 1 = 17 \\
\sum_{i=1}^{d} |a_{i,3}| = 4 + 10 + 2 + 8 = 24 \\
\sum_{i=1}^{d} |a_{i,4}| = 3 + 7 + 9 + 7 = 26
\]

then
\[
\|A\|_1 = \max \left\{ \sum_{i=1}^{d} |a_{ij}| : j = 1, \ldots, d \right\} = \max\{21, 17, 24, 26\} = 26.
\]

7. Prove or disprove whether \( \|A^2\|_\infty = \|A\|_2^2 \) holds in general for matrices \( A \in \mathbb{R}^{4 \times 4} \).

This is false. While the matrix \( A \) defined above could demonstrate that \( \|A^2\|_\infty \neq \|A\|_2^2 \), an easier example is
\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
for which \( A^2 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \).

In this case \( \|A\|_\infty^2 = 1 \) and \( \|A^2\|_\infty = 0 \).