

1. [Thomas Problem 1.2.1] Write a code to approximately solve the boundary value problem

$$\begin{cases} v_t = \nu v_{xx}, & x \in (0, 1), \quad t > 0 \\ v(x, 0) = f(x), & x \in [0, 1] \\ v(0, t) = a(t), & t \geq 0 \\ v(1, t) = b(t), & t \geq 0 \end{cases}$$

Use $f(x) = \sin 2\pi x$, $a(t) = b(t) = 0$, $K = 10$ and $\nu = 1/6$. Find solutions at $t = 0.06$, $t = 0.1$, $t = 0.9$ and $t = 50.0$. For the first three values of t , use $\Delta t = 0.02$. To speed the solution of the last value of t , you might use a larger value for Δt . Determine how large you can choose Δt and still get results that might still be correct. Compare and contrast your solutions to the exact solutions.

We compute the exact solution using separation of variables. First look for eigenfunctions $X(x)$ of the operator $\partial^2/\partial x^2$ such that $X(0) = X(1) = 0$. Solving

$$X''(x) = \lambda X(x)$$

yields

$$X(x) = A \exp(\sqrt{\lambda}x) + B \exp(-\sqrt{\lambda}x).$$

Now $X(0) = 0$ implies $B = -A$ so that

$$X(x) = A \exp(\sqrt{\lambda}x) - A \exp(-\sqrt{\lambda}x),$$

and on the other hand $X(1) = 0$ implies $\exp(\sqrt{\lambda}) = \exp(-\sqrt{\lambda})$. Therefore $\exp(2\sqrt{\lambda}) = 1$ implies $2\sqrt{\lambda} = 2n\pi i$ and we obtain eigenfunctions $X_n(x)$ with eigenvalues λ_n given by

$$X_n(x) = \sin(n\pi x) \quad \text{and} \quad \lambda_n = -n^2\pi^2 \quad \text{for} \quad n \in \mathbf{N}.$$

Consider a solution of the form

$$v(x, t) = \sum_{n \in \mathbf{N}} T_n(t) X_n(x).$$

Plugging this into the differential equation obtains

$$\sum_{n \in \mathbf{N}} T'_n(t) X_n(x) = \sum_{n \in \mathbf{N}} \nu \lambda_n T_n(t) X_n(x)$$

and since the eigenfunctions $X_n(x)$ are linearly independent, it follows that

$$T'_n(t) = \nu \lambda_n T_n(t) \quad \text{for all} \quad n \in \mathbf{N}.$$

Solving this system yields that $T_n(t) = T_n(0) \exp(-\nu n^2 \pi^2 t)$ and consequently

$$v(x, t) = \sum_{n \in \mathbf{N}} T_n(0) \exp(-\nu n^2 \pi^2 t) \sin(n\pi x).$$

The initial condition

$$v(x, 0) = \sum_{n \in \mathbf{N}} T_n(0) \sin(n\pi x) = f(x) = \sin(2\pi x)$$

implies

$$T_n(0) = \begin{cases} 1 & \text{if } n = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the exact solution is

$$v(x, t) = \exp(-\nu 4\pi^2 t) \sin(2\pi x).$$

We use the finite difference scheme

$$\begin{cases} u_k^{n+1} = u_k^n + \rho \delta^2 u_k^n & \text{for } n = 0, \dots, N-1 \text{ and } k = 1, \dots, K-1 \\ u_k^0 = f(x_k) & \text{for } k = 0, \dots, K \\ u_0^n = a(t_n) & \text{for } n = 1, \dots, N \\ u_K^n = b(t_n) & \text{for } n = 1, \dots, N \end{cases}$$

where $x_k = k\Delta x$, $t_n = n\Delta t$, $\Delta x = 1/K$, $\Delta t = T/N$ and $\rho = \nu\Delta t/\Delta x^2$. We chose the values of K , N and T given in the table

K	N	T	Δx	Δt	ρ
10	3	0.06	0.1	0.02	0.333333
10	5	0.1	0.1	0.02	0.333333
10	45	0.9	0.1	0.02	0.333333
10	2000	50	0.1	0.025	0.416667
10	1750	50	0.1	0.0285714	0.47619
10	1500	50	0.1	0.0333333	0.555556

In the following program, and all other programs included in this homework assignment, the output will be displayed in four columns. This output describes the approximation obtained by the numerical scheme, the exact solution and the error between the approximation and the exact solution. Specifically we have

1. The column `xk` gives the value of x_k .
2. The column `uk` gives the approximation u_k^N for $v(x_k, T)$.
3. The column `vk` gives the exact value of $v(x_k, T)$.
4. The column `vk-uk` gives the error $u_k^N - v(x_k, T)$.

The program is

```

1 /* prog1.c -- Thomas Problem 1.2.1
2    Written May 2013 by Eric Olson */
3
4 #include <stdio.h>
5 #include <math.h>

```

```

6 #define NU (1.0/6.0)
7
8 double dt,dx,rho;
9
10 double v(double x,double t){
11     return exp(-NU*4*M_PI*M_PI*t)*sin(2*M_PI*x);
12 }
13 double f(double x){
14     return sin(2*M_PI*x);
15 }
16 double a(double x){
17     return 0.0;
18 }
19 double b(double x){
20     return 0.0;
21 }
22 void doinit(int K,double u[K+1]){
23     int k;
24     for(k=0;k<=K;k++){
25         double xk=k*dx;
26         u[k]=f(xk);
27     }
28 }
29 void doprint(int K,double u[K+1],double tn){
30     int k;
31     printf("#%4s %23s %23s %23s\n","xk","uk","vk","uk-vk");
32     for(k=0;k<=K;k++){
33         double xk=k*dx;
34         double vk=v(xk,tn);
35         printf("%5g %23.14e %23.14e %23.14e\n",xk,u[k],vk,u[k]-vk);
36     }
37 }
38 void dosolve(int K,double u[K+1],int N,double T){
39     int n;
40     for(n=0;;n++){
41         double tn=n*dt;
42         u[0]=a(tn); u[K]=b(tn);
43         if(n>=N) break;
44         int k;
45         double w[K+1];
46         for(k=1;k<K;k++) w[k]=u[k-1]-2*u[k]+u[k+1];
47         for(k=1;k<K;k++) u[k]+=rho*w[k];
48     }
49 }
50 main(){

```

```

51     printf("# prog1 -- Thomas Problem 1.2.1\n"
52           "# Written May 2013 by Eric Olson\n#\n");
53     int KK[]={10,10,10,10,10,10};
54     int NN[]={3,5,45,2000,1750,1500};
55     double TT[]={0.06,0.1,0.9,50,50,50};
56     int J=sizeof(TT)/sizeof(double);
57     int j;
58     for(j=0;j<J;j++){
59         int K=KK[j],N=NN[j];
60         double T=TT[j],u[K+1];
61         dx=1.0/K; dt=T/N;
62         rho=NU*dt/dx/dx;
63         printf("#\tK=%d N=%d T=%g\n"
64               "\tdx=%g dt=%g rho=%g\n",
65               K,N,T,dx,dt,rho);
66         doinit(K,u);
67         dosolve(K,u,N,T);
68         dprint(K,u,T);
69         printf("\n\n");
70     }
71     return 0;
72 }

```

The output is

```

# prog1 -- Thomas Problem 1.2.1
# Written May 2013 by Eric Olson
#
# K=10 N=3 T=0.06
# dx=0.1 dt=0.02 rho=0.333333
#   xk          uk          vk          uk-vk
0   0.000000000000000e+00  0.000000000000000e+00  0.000000000000000e+00
0.1  3.90643644141615e-01  3.96064662853158e-01  -5.42101871154310e-03
0.2  6.32074693710251e-01  6.40846086239177e-01  -8.77139252892589e-03
0.3  6.32074693710251e-01  6.40846086239177e-01  -8.77139252892589e-03
0.4  3.90643644141615e-01  3.96064662853158e-01  -5.42101871154316e-03
0.5  1.27574455629128e-17  8.25170928945244e-17  -6.97596473316116e-17
0.6  -3.90643644141615e-01  -3.96064662853158e-01  5.42101871154332e-03
0.7  -6.32074693710251e-01  -6.40846086239177e-01  8.77139252892589e-03
0.8  -6.32074693710251e-01  -6.40846086239177e-01  8.77139252892600e-03
0.9  -3.90643644141615e-01  -3.96064662853158e-01  5.42101871154316e-03
1   0.000000000000000e+00  -1.65034185789049e-16  1.65034185789049e-16

# K=10 N=5 T=0.1
# dx=0.1 dt=0.02 rho=0.333333
#   xk          uk          vk          uk-vk
0   0.000000000000000e+00  0.000000000000000e+00  0.000000000000000e+00
0.1  2.97501263268220e-01  3.04413823505703e-01  -6.91256023748293e-03
0.2  4.81367155664011e-01  4.92551913077540e-01  -1.11847574135285e-02
0.3  4.81367155664011e-01  4.92551913077540e-01  -1.11847574135284e-02

```

0.4	2.97501263268220e-01	3.04413823505703e-01	-6.91256023748299e-03
0.5	7.58941520739856e-18	6.34223300095584e-17	-5.58329148021599e-17
0.6	-2.97501263268220e-01	-3.04413823505703e-01	6.91256023748310e-03
0.7	-4.81367155664011e-01	-4.92551913077540e-01	1.11847574135284e-02
0.8	-4.81367155664011e-01	-4.92551913077540e-01	1.11847574135285e-02
0.9	-2.97501263268220e-01	-3.04413823505703e-01	6.91256023748299e-03
1	0.00000000000000e+00	-1.26844660019117e-16	1.26844660019117e-16

K=10 N=45 T=0.9

dx=0.1 dt=0.02 rho=0.333333

#	xk	uk	vk	uk-vk
0	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00
0.1	1.28129710652919e-03	1.57554150100023e-03	-2.94244394471033e-04	-2.94244394471033e-04
0.2	2.07318226805113e-03	2.54927969930439e-03	-4.76097431253263e-04	-4.76097431253263e-04
0.3	2.07318226805113e-03	2.54927969930439e-03	-4.76097431253263e-04	-4.76097431253263e-04
0.4	1.28129710652919e-03	1.57554150100023e-03	-2.94244394471033e-04	-2.94244394471033e-04
0.5	2.62738974131952e-32	3.28252218869158e-19	-3.28252218869132e-19	-3.28252218869132e-19
0.6	-1.28129710652919e-03	-1.57554150100023e-03	2.94244394471034e-04	2.94244394471034e-04
0.7	-2.07318226805113e-03	-2.54927969930439e-03	4.76097431253263e-04	4.76097431253263e-04
0.8	-2.07318226805113e-03	-2.54927969930439e-03	4.76097431253263e-04	4.76097431253263e-04
0.9	-1.28129710652919e-03	-1.57554150100023e-03	2.94244394471033e-04	2.94244394471033e-04
1	0.00000000000000e+00	-6.56504437738316e-19	6.56504437738316e-19	6.56504437738316e-19

K=10 N=2000 T=50

dx=0.1 dt=0.025 rho=0.416667

#	xk	uk	vk	uk-vk
0	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00
0.1	1.59841078802718e-151	7.79939687131992e-144	-7.79939671147884e-144	-7.79939671147884e-144
0.2	2.58628298301247e-151	1.261968892295452e-143	-1.261968889709169e-143	-1.261968889709169e-143
0.3	2.58628298301247e-151	1.261968892295452e-143	-1.261968889709169e-143	-1.261968889709169e-143
0.4	1.59841078802718e-151	7.79939687131992e-144	-7.79939671147884e-144	-7.79939671147884e-144
0.5	1.42629035223595e-171	1.62494566295247e-159	-1.62494566295104e-159	-1.62494566295104e-159
0.6	-1.59841078802718e-151	-7.79939687131993e-144	7.79939671147885e-144	7.79939671147885e-144
0.7	-2.58628298301247e-151	-1.261968892295452e-143	1.261968889709169e-143	1.261968889709169e-143
0.8	-2.58628298301247e-151	-1.261968892295452e-143	1.261968889709169e-143	1.261968889709169e-143
0.9	-1.59841078802718e-151	-7.79939687131992e-144	7.79939671147885e-144	7.79939671147885e-144
1	0.00000000000000e+00	-3.24989132590493e-159	3.24989132590493e-159	3.24989132590493e-159

K=10 N=1750 T=50

dx=0.1 dt=0.0285714 rho=0.47619

#	xk	uk	vk	uk-vk
0	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00
0.1	-2.10725407775984e-54	7.79939687131992e-144	-2.10725407775984e-54	-2.10725407775984e-54
0.2	-4.00823544428607e-54	1.261968892295452e-143	-4.00823544428607e-54	-4.00823544428607e-54
0.3	-5.51686279850709e-54	1.261968892295452e-143	-5.51686279850709e-54	-5.51686279850709e-54
0.4	-6.48546118376690e-54	7.79939687131992e-144	-6.48546118376690e-54	-6.48546118376690e-54
0.5	-6.81921744149448e-54	1.62494566295247e-159	-6.81921744149448e-54	-6.81921744149448e-54
0.6	-6.48546118376690e-54	-7.79939687131993e-144	6.48546118376690e-54	6.48546118376690e-54
0.7	-5.51686279850709e-54	-1.261968892295452e-143	5.51686279850709e-54	5.51686279850709e-54
0.8	-4.00823544428607e-54	-1.261968892295452e-143	4.00823544428607e-54	4.00823544428607e-54
0.9	-2.10725407775984e-54	-7.79939687131992e-144	2.10725407775984e-54	2.10725407775984e-54
1	0.00000000000000e+00	-3.24989132590493e-159	3.24989132590493e-159	3.24989132590493e-159

```

# K=10 N=1500 T=50
# dx=0.1 dt=0.0333333 rho=0.555556
#  xk          uk          vk          uk-vk
  0    0.00000000000000e+00  0.00000000000000e+00  0.00000000000000e+00
0.1    1.21019810833957e+84  7.79939687131992e-144  1.21019810833957e+84
0.2   -2.30193359388883e+84  1.26196892295452e-143 -2.30193359388883e+84
0.3    3.16833978075383e+84  1.26196892295452e-143  3.16833978075383e+84
0.4   -3.72460679475733e+84  7.79939687131992e-144 -3.72460679475733e+84
0.5    3.91628334482851e+84  1.62494566295247e-159  3.91628334482851e+84
0.6   -3.72460679475733e+84 -7.79939687131993e-144 -3.72460679475733e+84
0.7    3.16833978075383e+84 -1.26196892295452e-143  3.16833978075383e+84
0.8   -2.30193359388883e+84 -1.26196892295452e-143 -2.30193359388883e+84
0.9    1.21019810833957e+84 -7.79939687131992e-144  1.21019810833957e+84
  1    0.00000000000000e+00 -3.24989132590493e-159  3.24989132590493e-159

```

Note that the last computation is clearly wrong because rather than decaying, the solution increases. In general, the scheme is unstable and therefore unreliable when $\rho > 1/2$.

2. [Thomas Problem 1.3.1] Solve Problem 1.2.1 using the leapfrog scheme

$$u_k^{n+1} = u_k^{n-1} + \frac{2\nu\Delta t}{\Delta x^2}(u_{k+1}^n - 2u_k^n + u_{k-1}^n)$$

with $\Delta t = 0.02$. For convenience, use values from the exact solution at Δt to get the leapfrog scheme started.

The program is

```

1 /* prog2.c -- Thomas Problem 1.3.1
2    Written May 2013 by Eric Olson */
3
4 #include <stdio.h>
5 #include <math.h>
6 #define NU (1.0/6.0)
7
8 double dt,dx,rho;
9
10 double v(double x,double t){
11     return exp(-NU*4*M_PI*M_PI*t)*sin(2*M_PI*x);
12 }
13 double a(double x){
14     return 0.0;
15 }
16 double b(double x){
17     return 0.0;
18 }
19 void doinit(int K,double u[2][K+1]){
20     int n;
21     for(n=0;n<2;n++){
22         int k;
23         for(k=0;k<=K;k++){
24             double xk=k*dx, tn=n*dt;
25             u[n][k]=v(xk,tn);
26         }
27     }
28 }
29 void doprint(int K,double u[K+1],double tn){
30     int k;
31     printf("#%4s %23s %23s %23s\n","xk","uk","vk","uk-vk");
32     for(k=0;k<=K;k++){
33         double xk=k*dx;
34         double vk=v(xk,tn);
35         printf("%5g %23.14e %23.14e %23.14e\n",xk,u[k],vk,u[k]-vk);
36     }
37 }

```

```

38 void dosolve(int K,double u[2][K+1],int N,double T){
39     int n;
40     for(n=1;;n++){
41         double tn=n*dt;
42         u[1][0]=a(tn); u[1][K]=b(tn);
43         if(n>=N) break;
44         int k;
45         double w[K+1];
46         for(k=1;k<K;k++) w[k]=u[1][k-1]-2*u[1][k]+u[1][k+1];
47         for(k=1;k<K;k++) {
48             double tmp=u[1][k];
49             u[1][k]=u[0][k]+rho*w[k];
50             u[0][k]=tmp;
51         }
52         u[0][0]=u[1][0]; u[0][K]=u[1][K];
53     }
54 }
55 main(){
56     printf("# prog2 -- Thomas Problem 1.3.1\n"
57           "# Written May 2013 by Eric Olson\n#\n");
58     int KK[]={10,10,10,10};
59     int NN[]={3,5,45,2500};
60     double TT[]={0.06,0.1,0.9,50};
61     int J=sizeof(TT)/sizeof(double);
62     int j;
63     for(j=0;j<J;j++){
64         int K=KK[j],N=NN[j];
65         double T=TT[j],u[2][K+1];
66         dx=1.0/K; dt=T/N;
67         rho=2*NU*dt/dx/dx;
68         printf("#\tK=%d N=%d T=%g\n"
69               "#\tdx=%g dt=%g rho=%g\n",
70               K,N,T,dx,dt,rho);
71         doinit(K,u);
72         dosolve(K,u,N,T);
73         doprint(K,u[1],T);
74         printf("\n\n");
75     }
76     return 0;
77 }

```

The output is

```

# prog2 -- Thomas Problem 1.3.1
# Written May 2013 by Eric Olson
#
# K=10 N=3 T=0.06

```



```

# dx=0.1 dt=0.02 rho=0.666667
# xk uk vk uk-vk
0 0.00000000000000e+00 0.00000000000000e+00 0.00000000000000e+00
0.1 3.99047602882670e-01 3.96064662853158e-01 2.98294002951238e-03
0.2 6.45672584593332e-01 6.40846086239177e-01 4.82649835415450e-03
0.3 6.45672584593331e-01 6.40846086239177e-01 4.82649835415372e-03
0.4 3.99047602882670e-01 3.96064662853158e-01 2.98294002951266e-03
0.5 3.44656492339396e-16 8.25170928945244e-17 2.62139399444871e-16
0.6 -3.99047602882671e-01 -3.96064662853158e-01 -2.98294002951299e-03
0.7 -6.45672584593331e-01 -6.40846086239177e-01 -4.82649835415372e-03
0.8 -6.45672584593331e-01 -6.40846086239177e-01 -4.82649835415405e-03
0.9 -3.99047602882671e-01 -3.96064662853158e-01 -2.98294002951272e-03
1 0.00000000000000e+00 -1.65034185789049e-16 1.65034185789049e-16

```

```

# K=10 N=5 T=0.1
# dx=0.1 dt=0.02 rho=0.666667
# xk uk vk uk-vk
0 0.00000000000000e+00 0.00000000000000e+00 0.00000000000000e+00
0.1 3.08661764863157e-01 3.04413823505703e-01 4.24794135745360e-03
0.2 4.99425226576122e-01 4.92551913077540e-01 6.87331349858250e-03
0.3 4.99425226576117e-01 4.92551913077540e-01 6.87331349857701e-03
0.4 3.08661764863159e-01 3.04413823505703e-01 4.24794135745582e-03
0.5 2.25554475130889e-15 6.34223300095584e-17 2.19212242129933e-15
0.6 -3.08661764863162e-01 -3.04413823505703e-01 -4.24794135745893e-03
0.7 -4.99425226576116e-01 -4.92551913077540e-01 -6.87331349857678e-03
0.8 -4.99425226576120e-01 -4.92551913077540e-01 -6.87331349858034e-03
0.9 -3.08661764863159e-01 -3.04413823505703e-01 -4.24794135745582e-03
1 0.00000000000000e+00 -1.26844660019117e-16 1.26844660019117e-16

```

```

# K=10 N=45 T=0.9
# dx=0.1 dt=0.02 rho=0.666667
# xk uk vk uk-vk
0 0.00000000000000e+00 0.00000000000000e+00 0.00000000000000e+00
0.1 4.61343688561026e+02 1.57554150100023e-03 4.61342113019525e+02
0.2 -1.09542685827749e+03 2.54927969930439e-03 -1.09542940755719e+03
0.3 1.96646303659455e+03 2.54927969930439e-03 1.96646048731485e+03
0.4 -2.98546825065064e+03 1.57554150100023e-03 -2.98546982619214e+03
0.5 3.91067459871360e+03 3.28252218869158e-19 3.91067459871360e+03
0.6 -4.44089303435913e+03 -1.57554150100023e-03 -4.44089145881763e+03
0.7 4.32364338351472e+03 -2.54927969930439e-03 4.32364593279442e+03
0.8 -3.45023526760994e+03 -2.54927969930439e-03 -3.45023271833024e+03
0.9 1.91808811081291e+03 -1.57554150100023e-03 1.91808968635441e+03
1 0.00000000000000e+00 -6.56504437738316e-19 6.56504437738316e-19

```

```

# K=10 N=2500 T=50
# dx=0.1 dt=0.02 rho=0.666667
# xk uk vk uk-vk
0 0.00000000000000e+00 0.00000000000000e+00 0.00000000000000e+00
0.1 -inf 7.79939687131992e-144 -inf
0.2 inf 1.26196892295452e-143 inf
0.3 -inf 1.26196892295452e-143 -inf
0.4 inf 7.79939687131992e-144 inf

```

0.5		-inf	1.62494566295247e-159		-inf
0.6		inf	-7.79939687131993e-144		inf
0.7		-inf	-1.26196892295452e-143		-inf
0.8		inf	-1.26196892295452e-143		inf
0.9		-inf	-7.79939687131992e-144		-inf
1	0.00000000000000e+00		-3.24989132590493e-159	3.24989132590493e-159	

Note that this method oscillates between $\pm\infty$ at $T = 50$ which is a noticeably poor result as the solution is supposed to converge to 0.

3. [Thomas Problem 1.3.2] Use $\Delta t = \Delta x^2/(6\nu)$ for the solution to Problem 1.2.1.

The program is

```
1 /* prog3.c -- Thomas Problem 1.3.2
2    Written May 2013 by Eric Olson */
3
4 #include <stdio.h>
5 #include <math.h>
6 #define NU (1.0/6.0)
7
8 double dt,dx,rho;
9
10 double v(double x,double t){
11     return exp(-NU*4*M_PI*M_PI*t)*sin(2*M_PI*x);
12 }
13 double f(double x){
14     return sin(2*M_PI*x);
15 }
16 double a(double x){
17     return 0.0;
18 }
19 double b(double x){
20     return 0.0;
21 }
22 void doinit(int K,double u[K+1]){
23     int k;
24     for(k=0;k<=K;k++){
25         double xk=k*dx;
26         u[k]=f(xk);
27     }
28 }
29 void doprint(int K,double u[K+1],double tn){
30     int k;
31     printf("#%4s %23s %23s %23s\n","xk","uk","vk","uk-vk");
32     for(k=0;k<=K;k++){
33         double xk=k*dx;
34         double vk=v(xk,tn);
35         printf("%5g %23.14e %23.14e %23.14e\n",xk,u[k],vk,u[k]-vk);
36     }
37 }
38 void dosolve(int K,double u[K+1],int N,double T){
39     int n;
40     for(n=0;;n++){
41         double tn=n*dt;
42         u[0]=a(tn); u[K]=b(tn);
```

```

43     if(n>=N) break;
44     int k;
45     double w[K+1];
46     for(k=1;k<K;k++) w[k]=u[k-1]-2*u[k]+u[k+1];
47     for(k=1;k<K;k++) u[k]+=rho*w[k];
48 }
49 }
50 main(){
51     printf("# prog3 -- Thomas Problem 1.3.2\n"
52           "# Written May 2013 by Eric Olson\n#\n");
53     int KK[]={10,10,10,10,10,10};
54     double TT[]={0.06,0.1,0.9,50,50,50};
55     int J=sizeof(TT)/sizeof(double);
56     int NN[J];
57     int j;
58     for(j=0;j<J;j++) NN[j]=KK[j]*KK[j]*TT[j]+0.5;
59     for(j=0;j<J;j++){
60         int K=KK[j],N=NN[j];
61         double T=TT[j],u[K+1];
62         dx=1.0/K; dt=T/N;
63         rho=NU*dt/dx/dx;
64         printf("#\tK=%d N=%d T=%g\n"
65               "#\tdx=%g dt=%g rho=%g\n",
66               K,N,T,dx,dt,rho);
67         doinit(K,u);
68         dosolve(K,u,N,T);
69         doprint(K,u,T);
70         printf("\n\n");
71     }
72     return 0;
73 }

```

The output is

```

# prog3 -- Thomas Problem 1.3.2
# Written May 2013 by Eric Olson
#
# K=10 N=6 T=0.06
# dx=0.1 dt=0.01 rho=0.166667
#   xk                uk                vk                uk-vk
0    0.000000000000000e+00  0.000000000000000e+00  0.000000000000000e+00  0.000000000000000e+00
0.1  3.96111414113491e-01  3.96064662853158e-01  4.67512603332665e-05
0.2  6.40921731367413e-01  6.40846086239177e-01  7.56451282361015e-05
0.3  6.40921731367413e-01  6.40846086239177e-01  7.56451282361015e-05
0.4  3.96111414113491e-01  3.96064662853158e-01  4.67512603332110e-05
0.5  4.65755183263568e-18  8.25170928945244e-17  -7.78595410618887e-17
0.6  -3.96111414113491e-01  -3.96064662853158e-01  -4.67512603331555e-05
0.7  -6.40921731367413e-01  -6.40846086239177e-01  -7.56451282362125e-05
0.8  -6.40921731367413e-01  -6.40846086239177e-01  -7.56451282361015e-05

```

```

0.9 -3.96111414113491e-01 -3.96064662853158e-01 -4.67512603332110e-05
1 0.00000000000000e+00 -1.65034185789049e-16 1.65034185789049e-16

```

```
# K=10 N=10 T=0.1
```

```
# dx=0.1 dt=0.01 rho=0.166667
```

```

# xk uk vk uk-vk
0 0.00000000000000e+00 0.00000000000000e+00 0.00000000000000e+00
0.1 3.04473713936096e-01 3.04413823505703e-01 5.98904303928949e-05
0.2 4.92648817829516e-01 4.92551913077540e-01 9.69047519764898e-05
0.3 4.92648817829516e-01 4.92551913077540e-01 9.69047519765454e-05
0.4 3.04473713936096e-01 3.04413823505703e-01 5.98904303928394e-05
0.5 -2.13542652889124e-17 6.34223300095584e-17 -8.47765952984708e-17
0.6 -3.04473713936096e-01 -3.04413823505703e-01 -5.98904303927839e-05
0.7 -4.92648817829516e-01 -4.92551913077540e-01 -9.69047519766564e-05
0.8 -4.92648817829516e-01 -4.92551913077540e-01 -9.69047519766009e-05
0.9 -3.04473713936096e-01 -3.04413823505703e-01 -5.98904303928394e-05
1 0.00000000000000e+00 -1.26844660019117e-16 1.26844660019117e-16

```

```
# K=10 N=90 T=0.9
```

```
# dx=0.1 dt=0.01 rho=0.166667
```

```

# xk uk vk uk-vk
0 0.00000000000000e+00 0.00000000000000e+00 0.00000000000000e+00
0.1 1.57833344829746e-03 1.57554150100023e-03 2.79194729723635e-06
0.2 2.55379716492612e-03 2.54927969930439e-03 4.51746562172553e-06
0.3 2.55379716492611e-03 2.54927969930439e-03 4.51746562172292e-06
0.4 1.57833344829745e-03 1.57554150100023e-03 2.79194729722702e-06
0.5 -1.43150156272752e-17 3.28252218869158e-19 -1.46432678461444e-17
0.6 -1.57833344829748e-03 -1.57554150100023e-03 -2.79194729725391e-06
0.7 -2.55379716492614e-03 -2.54927969930439e-03 -4.51746562174634e-06
0.8 -2.55379716492613e-03 -2.54927969930439e-03 -4.51746562174244e-06
0.9 -1.57833344829747e-03 -1.57554150100023e-03 -2.79194729724480e-06
1 0.00000000000000e+00 -6.56504437738316e-19 6.56504437738316e-19

```

```
# K=10 N=5000 T=50
```

```
# dx=0.1 dt=0.01 rho=0.166667
```

```

# xk uk vk uk-vk
0 0.00000000000000e+00 0.00000000000000e+00 0.00000000000000e+00
0.1 -3.78571845702551e-53 7.79939687131992e-144 -3.78571845702551e-53
0.2 -7.20086441482590e-53 1.26196892295452e-143 -7.20086441482590e-53
0.3 -9.91113959233060e-53 1.26196892295452e-143 -9.91113959233060e-53
0.4 -1.16512433715679e-52 7.79939687131992e-144 -1.16512433715679e-52
0.5 -1.22508422706102e-52 1.62494566295247e-159 -1.22508422706102e-52
0.6 -1.16512433715679e-52 -7.79939687131993e-144 -1.16512433715679e-52
0.7 -9.91113959233060e-53 -1.26196892295452e-143 -9.91113959233060e-53
0.8 -7.20086441482590e-53 -1.26196892295452e-143 -7.20086441482590e-53
0.9 -3.78571845702551e-53 -7.79939687131992e-144 -3.78571845702551e-53
1 0.00000000000000e+00 -3.24989132590493e-159 3.24989132590493e-159

```

```
# K=10 N=5000 T=50
```

```
# dx=0.1 dt=0.01 rho=0.166667
```

```

# xk uk vk uk-vk

```

0	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00
0.1	-3.78571845702551e-53	7.79939687131992e-144	-3.78571845702551e-53
0.2	-7.20086441482590e-53	1.26196892295452e-143	-7.20086441482590e-53
0.3	-9.91113959233060e-53	1.26196892295452e-143	-9.91113959233060e-53
0.4	-1.16512433715679e-52	7.79939687131992e-144	-1.16512433715679e-52
0.5	-1.22508422706102e-52	1.62494566295247e-159	-1.22508422706102e-52
0.6	-1.16512433715679e-52	-7.79939687131993e-144	-1.16512433715679e-52
0.7	-9.91113959233060e-53	-1.26196892295452e-143	-9.91113959233060e-53
0.8	-7.20086441482590e-53	-1.26196892295452e-143	-7.20086441482590e-53
0.9	-3.78571845702551e-53	-7.79939687131992e-144	-3.78571845702551e-53
1	0.00000000000000e+00	-3.24989132590493e-159	3.24989132590493e-159

K=10 N=5000 T=50

dx=0.1 dt=0.01 rho=0.166667

#	xk	uk	vk	uk-vk
0	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00
0.1	-3.78571845702551e-53	7.79939687131992e-144	-3.78571845702551e-53	-3.78571845702551e-53
0.2	-7.20086441482590e-53	1.26196892295452e-143	-7.20086441482590e-53	-7.20086441482590e-53
0.3	-9.91113959233060e-53	1.26196892295452e-143	-9.91113959233060e-53	-9.91113959233060e-53
0.4	-1.16512433715679e-52	7.79939687131992e-144	-1.16512433715679e-52	-1.16512433715679e-52
0.5	-1.22508422706102e-52	1.62494566295247e-159	-1.22508422706102e-52	-1.22508422706102e-52
0.6	-1.16512433715679e-52	-7.79939687131993e-144	-1.16512433715679e-52	-1.16512433715679e-52
0.7	-9.91113959233060e-53	-1.26196892295452e-143	-9.91113959233060e-53	-9.91113959233060e-53
0.8	-7.20086441482590e-53	-1.26196892295452e-143	-7.20086441482590e-53	-7.20086441482590e-53
0.9	-3.78571845702551e-53	-7.79939687131992e-144	-3.78571845702551e-53	-3.78571845702551e-53
1	0.00000000000000e+00	-3.24989132590493e-159	3.24989132590493e-159	3.24989132590493e-159

Note that this choice of Δt results in significantly more accurate accurate solutions than the choices used in Problem 1.2.1.

4. [Thomas Problem 1.5.1] Solve the problem

$$\begin{cases} v_t + av_x = \nu v_{xx}, & x \in (0, 1), \quad t > 0 \\ v(x, 0) = f(x), & x \in [0, 1] \\ v(0, t) = v(1, t) = 0, & t \geq 0 \end{cases}$$

using the difference scheme

$$\begin{cases} u_k^{n+1} = u_k^n - R\delta_0 u_k^n + \rho\delta^2 u_k^n & \text{for } n = 0, \dots, N \text{ and } k = 1, \dots, K-1 \\ u_k^0 = f(x_k) & \text{for } k = 0, \dots, K \\ u_0^n = u_K^n = 0 & \text{for } n = 1, \dots, N \end{cases}$$

where $R = a\Delta t/(2\Delta x)$ and $\rho = \nu\Delta t/\Delta x^2$. Use $\nu = 1.0$, $a = 2$, $K = 20$ and $f(x) = \sin 4\pi x$. Find solutions at $t = 0.06$, $t = 0.1$, $t = 0.9$ using $\Delta t = 0.001$.

We first find the exact solution. We begin by finding eigenfunctions $X(x)$ so that

$$\nu X''(x) - aX'(x) = \lambda X(x) \quad \text{and} \quad X(0) = X(1) = 0.$$

The characteristic equation is

$$\nu r^2 - ar - \lambda = 0$$

from which we determine that

$$r_1 = \frac{a + \sqrt{a^2 + 4\nu\lambda}}{2\nu} \quad \text{and} \quad r_2 = \frac{a - \sqrt{a^2 + 4\nu\lambda}}{2\nu}.$$

It follows that

$$X(x) = A \exp(r_1 x) + B \exp(r_2 x).$$

Since $X(0) = 0$ we obtain $B = -A$. Consequently

$$X(x) = A \exp(r_1 x) - A \exp(r_2 x).$$

Now $X(1) = 0$ implies $\exp(r_1) = \exp(r_2)$ or that $\exp(r_1 - r_2) = 1$. It follows that

$$r_1 - r_2 = \frac{\sqrt{a^2 + 4\nu\lambda}}{\nu} = 2n\pi i$$

and so we obtain eigenfunction $X_n(x)$ with eigenvalues λ_n given by

$$X_n(x) = \exp(ax/2) \sin(n\pi x) \quad \text{and} \quad \lambda_n = -\nu n^2 \pi^2 - \frac{a^2}{4\nu} \quad \text{for } n \in \mathbf{N}.$$

Consider a solution of the form

$$v(x, t) = \sum_{n \in \mathbf{N}} T_n(t) X_n(x).$$

Plugging this into the differential equation obtains

$$\sum_{n \in \mathbf{N}} T'_n(t) X_n(x) = \sum_{n \in \mathbf{N}} \lambda_n T_n(t) X_n(x)$$

and since the eigenfunctions $X_n(x)$ are linearly independent, it follows that

$$T'_n(t) = \lambda_n T_n(t) \quad \text{for all } n \in \mathbf{N}.$$

Solving this system yields that

$$T_n(t) = T_n(0) \exp\left(-\nu n^2 \pi^2 t - \frac{a^2 t}{4\nu}\right)$$

and consequently

$$v(x, t) = \sum_{n \in \mathbf{N}} T_n(0) \exp\left(-\nu n^2 \pi^2 t - \frac{a^2 t}{4\nu}\right) \exp(ax/2) \sin(n\pi x).$$

The initial condition

$$v(x, 0) = \sum_{n \in \mathbf{N}} T_n(0) \exp(ax/2) \sin(n\pi x) = f(x) = \sin(4\pi x)$$

implies

$$\sum_{n \in \mathbf{N}} T_n(0) \sin(n\pi x) = \exp(-ax/2) \sin(4\pi x)$$

Multiplying by $\sin(m\pi x)$ and integrating yields that

$$\begin{aligned} T_m(0) &= 2 \int_0^1 \exp(-ax/2) \sin(4\pi x) \sin(m\pi x) dx \\ &= \frac{128am(-1 + (-1)^m \exp(-a/2))}{16m^4\pi^4 + (8a^2\pi^2 - 512\pi^4)m^2 + a^4 + 128a^2\pi^2 + 4096\pi^4}. \end{aligned}$$

Therefore, the exact solution is a complicated series. We turn to Maple to help with this series. The maple script

```

1 restart;
2 R1:=2*exp(-a*x/2)*sin(4*Pi*x)*sin(m*Pi*x);
3 a:=2;
4 R1;
5 R2:=int(R1,x=0..1) assuming m::integer;
6 R3:=collect(R2,m);
7 with(codegen):
8 Digits:=15;
9 R4:=[optimize([TM0=evalf(R3)])];

```


10 C(R4);

produces the output

$$R1 := 2 \exp\left(-\frac{a x}{2}\right) \sin(4 \text{ Pi } x) \sin(m \text{ Pi } x)$$

$$a := 2$$

$$2 \exp(-x) \sin(4 \text{ Pi } x) \sin(m \text{ Pi } x)$$

$$R2 := -\frac{16 m^2 \text{ Pi}^2 (-1 + \exp(-1)) (-1)^m}{1 + 32 \text{ Pi}^2 + 2 m^2 \text{ Pi}^2 + 256 \text{ Pi}^4 - 32 \text{ Pi}^4 m^2 + m^4 \text{ Pi}^4}$$

$$R3 := -\frac{16 m^2 \text{ Pi}^2 (-1 + \exp(-1)) (-1)^m}{m^4 \text{ Pi}^4 + (2 \text{ Pi}^2 - 32 \text{ Pi}^4) m^2 + 1 + 32 \text{ Pi}^2 + 256 \text{ Pi}^4}$$

Warning, the protected name MathML has been redefined and unprotected
Digits := 15

$$R4 := [t1 = (-1.)^m, t5 = m^2, t6 = t5^2,$$

$$TM0 = -\frac{157.913670417429 m (-1. + 0.367879441171442 t1)}{97.4090910340020 t6 - 3097.35170428588 t5 + 25253.5546455394}]$$

```
t1 = pow(-1.0,1.0*m);
t5 = m*m;
t6 = t5*t5;
TM0 = -0.157913670417429E3*m*(-1.0+0.367879441171442*t1)/(
0.97409091034002E2*t6-0.309735170428588E4*t5+0.252535546455394E5);
```

We replace $\text{pow}(-1.0, 1.0*m)$ with $(1-2*(m\%2))$ to obtain the exact solution which, for convenience, we keep in a separate file of utility programs

```
1 /* util4.c -- Utility routines for prog4.c, prog5.c and prog6.c
2    Written May 2013 by Eric Olson */
3
4 #include <stdio.h>
5 #include <math.h>
6 #include "util4.h"
7
8 double dt,dx,R,rho,nu;
9
10 static double Tn0(int m){
```

```

11     double t1,t5,t6,TM0;
12     t1 = 1-2*(m%2);
13     t5 = m*m;
14     t6 = t5*t5;
15     TM0 = -0.157913670417429E3*m*(-1.0+0.367879441171442*t1)/(
16 0.97409091034002E2*t6-0.309735170428588E4*t5+0.252535546455394E5);
17     return TM0;
18 }
19 static double v(double x,double t){
20     int m;
21     double r=0;
22     for(m=100;m>=0;m--){
23         r+=Tn0(m)*exp(-t*(nu*m*m*M_PI*M_PI+A*A/4/nu))
24             *exp(A*x/2)*sin(m*M_PI*x);
25     }
26     return r;
27 }
28 static double f(double x){
29     return sin(4*M_PI*x);
30 }
31 void doinit(int K,double u[K+1]){
32     int k;
33     for(k=0;k<=K;k++){
34         double xk=k*dx;
35         u[k]=f(xk);
36     }
37 }
38 void doprint(int K,double u[K+1],double tn){
39     int k;
40     printf("#%4s %23s %23s %23s\n","xk","uk","vk","vk-uk");
41     for(k=0;k<=K;k++){
42         double xk=k*dx;
43         double vk=v(xk,tn);
44         printf("%5g %23.14e %23.14e %23.14e\n",xk,u[k],vk,vk-u[k]);
45     }
46 }
47 void dosolve(int K,double u[K+1],int N,double T){
48     int n;
49     for(n=0;;n++){
50         double tn=n*dt;
51         u[0]=0; u[K]=0;
52         if(n>=N) break;
53         int k;
54         double w[K+1];
55         for(k=1;k<K;k++) w[k]=-R*(u[k+1]-u[k-1])

```



```

24         K,N,T,A,nu,dx,dt,R,rho);
25     doinit(K,u);
26     dosolve(K,u,N,T);
27     doprint(K,u,T);
28     printf("\n\n");
29 }
30 return 0;
31 }

```

The output is

```

# prog4 -- Thomas Problem 1.5.1
# Written May 2013 by Eric Olson
#
# K=20 N=60 T=0.06 A=2 nu=1
# dx=0.05 dt=0.001 R=0.02 rho=0.4
# xk          uk          vk          vk-uk
0      0.00000000000000e+00  0.00000000000000e+00  0.00000000000000e+00
0.05   1.46623413685960e-03  1.52411165668599e-03  5.78775198263879e-05
0.1    2.96140439292754e-03  3.06944515101641e-03  1.08040758088868e-04
0.15   4.37029190625183e-03  4.50864167696056e-03  1.38349770708739e-04
0.2    5.59138981384089e-03  5.73237428059674e-03  1.40984466755851e-04
0.25   6.55255511765887e-03  6.66730132115240e-03  1.14746203493528e-04
0.3    7.22059773237988e-03  7.28616730215376e-03  6.55695697738866e-05
0.35   7.60270056556878e-03  7.60773429567998e-03  5.03373011120304e-06
0.4    7.73950472930428e-03  7.68679002453340e-03  -5.27147047708846e-05
0.45   7.69167545470254e-03  7.59696280546681e-03  -9.47126492357297e-05
0.5    7.52325815947266e-03  7.41077410707977e-03  -1.12484052392886e-04
0.55   7.28575688199937e-03  7.18181728070921e-03  -1.03939601290156e-04
0.6    7.00649166875928e-03  6.93309534797398e-03  -7.33963207852966e-05
0.65   6.68357370731331e-03  6.65370952611011e-03  -2.98641812032010e-05
0.7    6.28815293167090e-03  6.30386451572086e-03  1.57115840499523e-05
0.75   5.77290814513533e-03  5.82620604176428e-03  5.32978966289507e-05
0.8    5.08447201949534e-03  5.16032546934095e-03  7.58534498456163e-05
0.85   4.17684973176715e-03  4.25703694928099e-03  8.01872175138427e-05
0.9    3.02292757475456e-03  3.08960309001853e-03  6.66755152639743e-05
0.95   1.62173455201314e-03  1.66008307516332e-03  3.83485231501752e-05
1      0.00000000000000e+00  1.37079784627557e-18  1.37079784627557e-18

# K=20 N=100 T=0.1 A=2 nu=1
# dx=0.05 dt=0.001 R=0.02 rho=0.4
# xk          uk          vk          vk-uk
0      0.00000000000000e+00  0.00000000000000e+00  0.00000000000000e+00
0.05   6.17817145957152e-04  6.24271147151420e-04  6.45400119426813e-06
0.1    1.27607371097953e-03  1.28883246373243e-03  1.27587527528994e-05
0.15   1.95343279265092e-03  1.97160491314917e-03  1.81721204982424e-05
0.2    2.62678834145056e-03  2.64883289500422e-03  2.20445535536657e-05
0.25   3.27253689609076e-03  3.29646327789337e-03  2.39263818026130e-05
0.3    3.86781273108086e-03  3.89145589804292e-03  2.36431669620590e-05
0.35   4.39155461145447e-03  4.41287807446811e-03  2.13234630136363e-05
0.4    4.82530184617355e-03  4.84267452746652e-03  1.73726812929620e-05
0.45   5.15366199665063e-03  5.16605981039609e-03  1.23978137454624e-05
0.5    5.36444430241762e-03  5.37154240064203e-03  7.09809822440916e-06

```

0.55	5.44850226944188e-03	5.45064622321818e-03	2.14395377629798e-06
0.6	5.39936726558439e-03	5.39743622815100e-03	-1.93103743339421e-06
0.65	5.21277623005713e-03	5.20797290977389e-03	-4.80332028323944e-06
0.7	4.88619836136673e-03	4.87981454722936e-03	-6.38381413736299e-06
0.75	4.41844975405689e-03	4.41165898743486e-03	-6.79076662202845e-06
0.8	3.80945696254658e-03	3.80317694700718e-03	-6.28001553939746e-06
0.85	3.06019825135991e-03	3.05504657453517e-03	-5.15167682474133e-06
0.9	2.17282307586214e-03	2.16916491087679e-03	-3.65816498534887e-06
0.95	1.15093261982319e-03	1.14899388963470e-03	-1.93873018848579e-06
1	0.00000000000000e+00	9.44092120664098e-19	9.44092120664098e-19

K=20 N=900 T=0.9 A=2 nu=1
dx=0.05 dt=0.001 R=0.02 rho=0.4

#	xk	uk	vk	vk-uk
0	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00
0.05	8.78925634351604e-08	9.00670446937724e-08	2.17448125861201e-09	2.17448125861201e-09
0.1	1.82530271892983e-07	1.87038305638134e-07	4.50803374515054e-09	4.50803374515054e-09
0.15	2.81924064740487e-07	2.88874817732295e-07	6.95075299180805e-09	6.95075299180805e-09
0.2	3.83739862589602e-07	3.93184445797730e-07	9.44458320812892e-09	9.44458320812892e-09
0.25	4.85328817686116e-07	4.97252951916199e-07	1.19241342300834e-08	1.19241342300834e-08
0.3	5.83769725772004e-07	5.98087514749589e-07	1.43177889775848e-08	1.43177889775848e-08
0.35	6.75923669986131e-07	6.92472769495930e-07	1.65490995097982e-08	1.65490995097982e-08
0.4	7.58500625937397e-07	7.77039087336565e-07	1.85384613991681e-08	1.85384613991681e-08
0.45	8.28137387438453e-07	8.48342434658105e-07	2.02050472196516e-08	2.02050472196516e-08
0.5	8.81485782099602e-07	9.02954752758228e-07	2.14689706586262e-08	2.14689706586262e-08
0.55	9.15309744009434e-07	9.37563387351391e-07	2.22536433419570e-08	2.22536433419570e-08
0.6	9.26589407305591e-07	9.49077684466671e-07	2.24882771610797e-08	2.24882771610797e-08
0.65	9.12629990918799e-07	9.34740466904159e-07	2.21104759853595e-08	2.21104759853595e-08
0.7	8.71172873421566e-07	8.92241725846328e-07	2.10688524247628e-08	2.10688524247628e-08
0.75	8.00505920654273e-07	8.19831518733828e-07	1.93255980795553e-08	1.93255980795553e-08
0.8	6.99569840919900e-07	7.16428770700332e-07	1.68589297804317e-08	1.68589297804317e-08
0.85	5.68057116337059e-07	5.81722446320546e-07	1.36653299834868e-08	1.36653299834868e-08
0.9	4.06499907363295e-07	4.16261404387515e-07	9.76149702421930e-09	9.76149702421930e-09
0.95	2.16343262724872e-07	2.21529183338179e-07	5.18592061330660e-09	5.18592061330660e-09
1	0.00000000000000e+00	1.82309683727663e-22	1.82309683727663e-22	1.82309683727663e-22

5. [Thomas Problem 1.5.2] Repeat Problem 1.5.1 with $\nu = 0.01$.

The program is

```

1 /* prog5.c -- Thomas Problem 1.5.2
2    Written May 2013 by Eric Olson */
3
4 #include <stdio.h>
5 #include <math.h>
6 #include "util4.h"
7
8 main(){
9     printf("# prog5 -- Thomas Problem 1.5.2\n"
10           "# Written May 2013 by Eric Olson\n#\n");
11     nu=0.01;
12     int KK[]={20,20,20};
13     double TT[]={0.06,0.1,0.9};
14     int NN[]={60,100,900};
15     int J=sizeof(TT)/sizeof(double);
16     int j;
17     for(j=0;j<J;j++){
18         int K=KK[j],N=NN[j];
19         double T=TT[j],u[K+1];
20         dx=1.0/K; dt=T/N;
21         R=A*dt/(2*dx); rho=nu*dt/dx/dx;
22         printf("#\tK=%d N=%d T=%g A=%g nu=%g\n"
23               "\tdx=%g dt=%g R=%g rho=%g\n",
24               K,N,T,A,nu,dx,dt,R,rho);
25         doinit(K,u);
26         dosolve(K,u,N,T);
27         doprint(K,u,T);
28         printf("\n\n");
29     }
30     return 0;
31 }

```

The output is

```

# prog5 -- Thomas Problem 1.5.2
# Written May 2013 by Eric Olson
#
# K=20 N=60 T=0.06 A=2 nu=0.01
# dx=0.05 dt=0.001 R=0.02 rho=0.004
#  xk                uk                vk                vk-uk
#  0    0.000000000000000e+00    0.000000000000000e+00    0.000000000000000e+00
0.05  -3.66160688906034e-02    1.30037164365576e-03    3.79164405342592e-02
0.1    1.41033226496769e-01    2.13488161551833e-03   -1.38898344881251e-01
0.15   5.20102633220556e-01    2.15588150061455e-03   -5.17946751719941e-01
0.2    8.53605901727816e-01    1.35343281898829e-03   -8.52252468908828e-01

```

0.25	9.21967139223049e-01	3.40188567416987e-05	-9.21933120366307e-01
0.3	6.56302514259971e-01	-1.29838915250333e-03	-6.57600903412474e-01
0.35	1.44253176488393e-01	-2.13485663611623e-03	-1.46388033124510e-01
0.4	-4.22051975922827e-01	-2.15588144584100e-03	4.19896094476986e-01
0.45	-8.27006946578637e-01	-1.35343281896978e-03	8.25653513759667e-01
0.5	-9.16053076685987e-01	-3.40188567416988e-05	9.16019057829245e-01
0.55	-6.55195588718864e-01	1.29838915250327e-03	6.56493977871367e-01
0.6	-1.44074257692972e-01	2.13485663611614e-03	1.46209114329089e-01
0.65	4.22066502486374e-01	2.15588144584090e-03	-4.19910621040533e-01
0.7	8.27107213876885e-01	1.35343281896972e-03	-8.25753781057915e-01
0.75	9.15274661543513e-01	3.40188567416989e-05	-9.15240642686771e-01
0.8	6.60449047432807e-01	-1.29838915249082e-03	-6.61747436585298e-01
0.85	1.15030577143323e-01	-2.13485659553884e-03	-1.17165433738862e-01
0.9	-2.95362045538957e-01	-2.15586099443615e-03	2.93206184544521e-01
0.95	-1.23939634661129e+00	-1.35163898679403e-03	1.23804470762450e+00
1	0.00000000000000e+00	-1.14739460755934e-18	-1.14739460755934e-18

K=20 N=100 T=0.1 A=2 nu=0.01
dx=0.05 dt=0.001 R=0.02 rho=0.004

#	xk	uk	vk	vk-uk
0	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00
0.05	-2.63084605264710e-02	2.21408302683046e-05	2.63306013567393e-02	2.63306013567393e-02
0.1	-7.02940031400941e-02	3.66012401000233e-05	7.03306043801941e-02	7.03306043801941e-02
0.15	4.17282994455560e-02	3.71973461843038e-05	-4.16911020993717e-02	-4.16911020993717e-02
0.2	3.52498465430278e-01	2.35918633298297e-05	-3.52474873566949e-01	-3.52474873566949e-01
0.25	7.09180810204167e-01	9.75219953555280e-07	-7.09179834984214e-01	-7.09179834984214e-01
0.3	8.99757619415463e-01	-2.20139234423312e-05	-8.99779633338905e-01	-8.99779633338905e-01
0.35	7.93457658961539e-01	-3.65944963071668e-05	-7.93494253457846e-01	-7.93494253457846e-01
0.4	4.01110907392467e-01	-3.71972153838459e-05	-4.01148104607851e-01	-4.01148104607851e-01
0.45	-1.39221499735007e-01	-2.35918624707464e-05	1.39197907872536e-01	1.39197907872536e-01
0.5	-6.24988951315169e-01	-9.75219951735159e-07	6.24987976095218e-01	6.24987976095218e-01
0.55	-8.71723674998390e-01	2.20139234423315e-05	8.71745688921832e-01	8.71745688921832e-01
0.6	-7.85322839148934e-01	3.65944963071652e-05	7.85359433645242e-01	7.85359433645242e-01
0.65	-3.99554002518141e-01	3.71972153838442e-05	3.99591199733525e-01	3.99591199733525e-01
0.7	1.42135464869813e-01	2.35918624707460e-05	-1.42111873007342e-01	-1.42111873007342e-01
0.75	6.14006671200832e-01	9.75219952839121e-07	-6.14005695980879e-01	-6.14005695980879e-01
0.8	9.13767658112682e-01	-2.20139228664701e-05	-9.13789672035549e-01	-9.13789672035549e-01
0.85	6.54793908186113e-01	-3.65943994078017e-05	-6.54830502585520e-01	-6.54830502585520e-01
0.9	7.16910068116009e-01	-3.71916940332391e-05	-7.16947259810043e-01	-7.16947259810043e-01
0.95	-7.03287965884016e-01	-2.34770324260008e-05	7.03264488851590e-01	7.03264488851590e-01
1	0.00000000000000e+00	-2.01080853901805e-20	-2.01080853901805e-20	-2.01080853901805e-20

K=20 N=900 T=0.9 A=2 nu=0.01
dx=0.05 dt=0.001 R=0.02 rho=0.004

#	xk	uk	vk	vk-uk
0	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00
0.05	-8.11561198052876e-05	1.11340650376665e-40	8.11561198052876e-05	8.11561198052876e-05
0.1	-2.60651946725292e-04	1.92263668825717e-40	2.60651946725292e-04	2.60651946725292e-04
0.15	4.21163581407451e-04	2.10483169779242e-40	-4.21163581407451e-04	-4.21163581407451e-04
0.2	-7.53110622485622e-04	1.56189542823066e-40	7.53110622485622e-04	7.53110622485622e-04
0.25	1.19319596362230e-03	4.71644314004358e-41	-1.19319596362230e-03	-1.19319596362230e-03
0.3	-1.23291234388256e-03	-7.72216104333435e-41	1.23291234388256e-03	1.23291234388256e-03
0.35	9.45781420422263e-04	-1.70871084303654e-40	-9.45781420422263e-04	-9.45781420422263e-04

0.4	1.22642083529880e-04	-1.98748635243617e-40	-1.22642083529880e-04
0.45	-8.56280089345500e-04	-1.50531340252608e-40	8.56280089345500e-04
0.5	1.80578185852223e-03	-4.47591518289244e-41	-1.80578185852223e-03
0.55	-2.75191311881464e-03	7.81310041861009e-41	2.75191311881464e-03
0.6	9.51019088111692e-04	1.71207503603697e-40	-9.51019088111692e-04
0.65	1.87194122487970e-03	1.98982993595506e-40	-1.87194122487970e-03
0.7	-2.41820931262331e-03	1.51047496818956e-40	2.41820931262331e-03
0.75	2.67723179539364e-03	4.62207356998431e-41	-2.67723179539364e-03
0.8	-3.21841620843348e-03	-7.43432877196221e-41	3.21841620843348e-03
0.85	-4.45160751947331e-03	-1.62539465301124e-40	4.45160751947331e-03
0.9	7.67405525760697e-03	-1.81563981832394e-40	-7.67405525760697e-03
0.95	1.93376654716871e-03	-1.20495500508405e-40	-1.93376654716871e-03
1	0.00000000000000e+00	-1.06147935096202e-55	-1.06147935096202e-55

6. [Thomas Problem 1.5.3] Repeat Problem 1.5.1 and 1.5.2 with $K=40$.

The program is

```

1 /* prog6.c -- Thomas Problem 1.5.3
2    Written May 2013 by Eric Olson */
3
4 #include <stdio.h>
5 #include <math.h>
6 #include "util4.h"
7
8 main(){
9     printf("# prog6 -- Thomas Problem 1.5.3\n"
10           "# Written May 2013 by Eric Olson\n#\n");
11     int KK[]={40,40,40,40,40,40,40,40,40,40,40,40};
12     double TT[]={0.06,0.06,0.1,0.1,0.9,0.9,0.06,0.06,0.1,0.1,0.9,0.9};
13     int NN[]={60,600,100,1000,900,9000,60,600,100,1000,900,9000};
14     double NU[]={1,1,1,1,1,1,0.01,0.01,0.01,0.01,0.01,0.01};
15     int J=sizeof(TT)/sizeof(double);
16     int j;
17     for(j=0;j<J;j++){
18         nu=NU[j];
19         int K=KK[j],N=NN[j];
20         double T=TT[j],u[K+1];
21         dx=1.0/K; dt=T/N;
22         R=A*dt/(2*dx); rho=nu*dt/dx/dx;
23         printf("#\tK=%d N=%d T=%g A=%g nu=%g\n"
24               "\tdx=%g dt=%g R=%g rho=%g\n",
25               K,N,T,A,nu,dx,dt,R,rho);
26         doinit(K,u);
27         dosolve(K,u,N,T);
28         doprint(K,u,T);
29         printf("\n\n");
30     }
31     return 0;
32 }

```

The output is

```

# prog6 -- Thomas Problem 1.5.3
# Written May 2013 by Eric Olson
#
# K=40 N=60 T=0.06 A=2 nu=1
# dx=0.025 dt=0.001 R=0.04 rho=1.6
#  xk                uk                vk                vk-uk
#  0      0.000000000000000e+00    0.000000000000000e+00    0.000000000000000e+00
0.025    3.40618843526207e+38    7.51325964408646e-04    -3.40618843526207e+38
0.05     -6.69593287459587e+38    1.52411165668599e-03    6.69593287459587e+38
0.075     9.59777492301421e+38    2.30226100030866e-03    -9.59777492301421e+38

```

0.1	-1.18881458358899e+39	3.06944515101641e-03	1.18881458358899e+39
0.125	1.34195715032897e+39	3.80980600342571e-03	-1.34195715032897e+39
0.15	-1.41354392853401e+39	4.50864167696056e-03	1.41354392853401e+39
0.175	1.40694679348159e+39	5.15303314832775e-03	-1.40694679348159e+39
0.2	-1.33313012778473e+39	5.73237428059674e-03	1.33313012778473e+39
0.225	1.20823219875280e+39	6.23877324061957e-03	-1.20823219875280e+39
0.25	-1.05072434780705e+39	6.66730132115240e-03	1.05072434780705e+39
0.275	8.78707067888556e+38	7.01607493260423e-03	-8.78707067888556e+38
0.3	-7.07782085954383e+38	7.28616730215376e-03	7.07782085954383e+38
0.325	5.49745762542403e+38	7.48135742991628e-03	-5.49745762542403e+38
0.35	-4.12141647441791e+38	7.60773429567998e-03	4.12141647441791e+38
0.375	2.98540847031290e+38	7.67318342211199e-03	-2.98540847031290e+38
0.4	-2.09318784617957e+38	7.68679002453340e-03	2.09318784617957e+38
0.425	1.42671749281719e+38	7.65819761716095e-03	-1.42671749281719e+38
0.45	-9.56514283709219e+37	7.59696280546681e-03	9.56514283709219e+37
0.475	6.50641063613344e+37	7.51194600149380e-03	-6.50641063613344e+37
0.5	-4.81558151198805e+37	7.41077410707977e-03	4.81558151198805e+37
0.525	4.30643230036509e+37	7.29940518146321e-03	-4.30643230036509e+37
0.55	-4.90526677423843e+37	7.18181728070921e-03	4.90526677423843e+37
0.575	6.65467202465366e+37	7.05983468601469e-03	-6.65467202465366e+37
0.6	-9.69890109030230e+37	6.93309534797398e-03	9.69890109030230e+37
0.625	1.42505538944490e+38	6.79915428360826e-03	-1.42505538944490e+38
0.65	-2.05375337744338e+38	6.65370952611011e-03	2.05375337744338e+38
0.675	2.87305381763018e+38	6.49093057685101e-03	-2.87305381763018e+38
0.7	-3.88551169948815e+38	6.30386451572086e-03	3.88551169948815e+38
0.725	5.06982405456966e+38	6.08489217268453e-03	-5.06982405456966e+38
0.75	-6.37259983923886e+38	5.82620604176428e-03	6.37259983923886e+38
0.775	7.70342740832074e+38	5.52028274222164e-03	-7.70342740832074e+38
0.8	-8.93554082258662e+38	5.16032546934095e-03	8.93554082258662e+38
0.825	9.91388096892215e+38	4.74065559951786e-03	-9.91388096892215e+38
0.85	-1.04711425942684e+39	4.25703694928099e-03	1.04711425942684e+39
0.875	1.04506304400283e+39	3.70692067879109e-03	-1.04506304400283e+39
0.9	-9.73278560852171e+38	3.08960309001853e-03	9.73278560852171e+38
0.925	8.26062340799465e+38	2.40629232384582e-03	-8.26062340799465e+38
0.95	-6.05860440947375e+38	1.66008307516332e-03	6.05860440947375e+38
0.975	3.24003291920665e+38	8.55840936326871e-04	-3.24003291920665e+38
1	0.00000000000000e+00	1.37079784627557e-18	1.37079784627557e-18

K=40 N=600 T=0.06 A=2 nu=1
dx=0.025 dt=0.0001 R=0.004 rho=0.16

#	xk	uk	vk	vk-uk
	0	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00
0.025	7.51996991590100e-04	7.51325964408646e-04	-6.71027181454726e-07	
0.05	1.52546528620609e-03	1.52411165668599e-03	-1.35362952010396e-06	
0.075	2.30427871554019e-03	2.30226100030866e-03	-2.01771523153371e-06	
0.1	3.07207866982140e-03	3.06944515101641e-03	-2.63351880499174e-06	
0.125	3.81297945043462e-03	3.80980600342571e-03	-3.17344700891077e-06	
0.15	4.5122550218488e-03	4.50864167696056e-03	-3.61382522431983e-06	
0.175	5.15696956335095e-03	5.15303314832775e-03	-3.93641502320543e-06	
0.2	5.73650386885673e-03	5.73237428059674e-03	-4.12958825998701e-06	
0.225	6.24296230756765e-03	6.23877324061957e-03	-4.18906694807635e-06	
0.25	6.67141949121091e-03	6.66730132115240e-03	-4.11817005851241e-06	
0.275	7.02000247817540e-03	7.01607493260423e-03	-3.92754557117063e-06	
0.3	7.28980170772069e-03	7.28616730215376e-03	-3.63440556692268e-06	

0.325	7.48461875045898e-03	7.48135742991628e-03	-3.26132054270095e-06
0.35	7.61056895884714e-03	7.60773429567998e-03	-2.83466316715851e-06
0.375	7.67556624052862e-03	7.67318342211199e-03	-2.38281841662656e-06
0.4	7.68872431863527e-03	7.68679002453340e-03	-1.93429410187684e-06
0.425	7.65971348888346e-03	7.65819761716095e-03	-1.51587172251156e-06
0.45	7.59811373733743e-03	7.59696280546681e-03	-1.15093187062851e-06
0.475	7.51280407310092e-03	7.51194600149380e-03	-8.58071607117220e-07
0.5	7.41142421200477e-03	7.41077410707977e-03	-6.50104924998854e-07
0.525	7.29993868550500e-03	7.29940518146321e-03	-5.33504041792289e-07
0.55	7.18232558263088e-03	7.18181728070921e-03	-5.08301921670583e-07
0.575	7.06040312458214e-03	7.05983468601469e-03	-5.68438567451722e-07
0.6	6.93379784668707e-03	6.93309534797398e-03	-7.02498713094253e-07
0.625	6.80004904335106e-03	6.79915428360826e-03	-8.94759742807276e-07
0.65	6.65483597459749e-03	6.65370952611011e-03	-1.12644848737978e-06
0.675	6.49230767249425e-03	6.49093057685101e-03	-1.37709564324191e-06
0.7	6.30549039323797e-03	6.30386451572086e-03	-1.62587751711615e-06
0.725	6.08674501878385e-03	6.08489217268453e-03	-1.85284609931571e-06
0.75	5.82824601029631e-03	5.82620604176428e-03	-2.03996853203464e-06
0.775	5.52245466561031e-03	5.52028274222164e-03	-2.17192338866461e-06
0.8	5.16256210004251e-03	5.16032546934095e-03	-2.23663070155703e-06
0.825	4.74288112146817e-03	4.74065559951786e-03	-2.22552195031089e-06
0.85	4.25917053117877e-03	4.25703694928099e-03	-2.13358189777527e-06
0.875	3.70887989211731e-03	3.70692067879109e-03	-1.95921332622794e-06
0.9	3.09130707626578e-03	3.08960309001853e-03	-1.70398624724710e-06
0.925	2.40766465778395e-03	2.40629232384582e-03	-1.37233393813184e-06
0.95	1.66105432445317e-03	1.66008307516332e-03	-9.71249289851133e-07
0.975	8.56350954073910e-04	8.55840936326871e-04	-5.10017747038980e-07
1	0.00000000000000e+00	1.37079784627557e-18	1.37079784627557e-18

K=40 N=100 T=0.1 A=2 nu=1
dx=0.025 dt=0.001 R=0.04 rho=1.6

#	xk	uk	vk	vk-uk
0	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00
0.025	3.10315866557873e+67	3.05821339524153e-04	-3.10315866557873e+67	-3.10315866557873e+67
0.05	-6.20430247910561e+67	6.24271147151420e-04	6.20430247910561e+67	6.20430247910561e+67
0.075	9.14760413452547e+67	9.52840265494388e-04	-9.14760413452547e+67	-9.14760413452547e+67
0.1	-1.17877382493598e+68	1.28883246373243e-03	1.17877382493598e+68	1.17877382493598e+68
0.125	1.40017580202047e+68	1.62940382814589e-03	-1.40017580202047e+68	-1.40017580202047e+68
0.15	-1.56985699496927e+68	1.97160491314917e-03	1.56985699496927e+68	1.56985699496927e+68
0.175	1.68250084277222e+68	2.31242439169945e-03	-1.68250084277222e+68	-1.68250084277222e+68
0.2	-1.73679868655905e+68	2.64883289500422e-03	1.73679868655905e+68	1.73679868655905e+68
0.225	1.73527319896549e+68	2.97782573685322e-03	-1.73527319896549e+68	-1.73527319896549e+68
0.25	-1.68376062024665e+68	3.29646327789337e-03	1.68376062024665e+68	1.68376062024665e+68
0.275	1.59064132042703e+68	3.60190779725900e-03	-1.59064132042703e+68	-1.59064132042703e+68
0.3	-1.46593115564962e+68	3.89145589804292e-03	1.46593115564962e+68	1.46593115564962e+68
0.325	1.32035081787015e+68	4.16256567163794e-03	-1.32035081787015e+68	-1.32035081787015e+68
0.35	-1.16447798448423e+68	4.41287807446811e-03	1.16447798448423e+68	1.16447798448423e+68
0.375	1.00806134429867e+68	4.64023221802582e-03	-1.00806134429867e+68	-1.00806134429867e+68
0.4	-8.59541998732333e+67	4.84267452746652e-03	8.59541998732333e+67	8.59541998732333e+67
0.425	7.25792337963791e+67	5.01846197303120e-03	-7.25792337963791e+67	-7.25792337963791e+67
0.45	-6.12050748585540e+67	5.16605981039609e-03	6.12050748585540e+67	6.12050748585540e+67
0.475	5.22006557865821e+67	5.28413446983959e-03	-5.22006557865821e+67	-5.22006557865821e+67
0.5	-4.57975850486211e+67	5.37154240064203e-03	4.57975850486211e+67	4.57975850486211e+67
0.525	4.21105830508991e+67	5.42731579928773e-03	-4.21105830508991e+67	-4.21105830508991e+67

0.55	-4.11552366241213e+67	5.45064622321818e-03	4.11552366241213e+67
0.575	4.28590280419018e+67	5.44086711423556e-03	-4.28590280419018e+67
0.6	-4.70636199641417e+67	5.39743622815100e-03	4.70636199641417e+67
0.625	5.35186405350948e+67	5.31991889364165e-03	-5.35186405350948e+67
0.65	-6.18694110515670e+67	5.20797290977389e-03	6.18694110515670e+67
0.675	7.16428993813855e+67	5.06133574665621e-03	-7.16428993813855e+67
0.7	-8.22373984658163e+67	4.87981454722936e-03	8.22373984658163e+67
0.725	9.29218007352843e+67	4.66327925134288e-03	-9.29218007352843e+67
0.75	-1.02849723287553e+68	4.41165898743486e-03	1.02849723287553e+68
0.775	1.11092105501525e+68	4.12494171343100e-03	-1.11092105501525e+68
0.8	-1.16689366763702e+68	3.80317694700718e-03	1.16689366763702e+68
0.825	1.18721057231520e+68	3.44648131457678e-03	-1.18721057231520e+68
0.85	-1.16387592434738e+68	3.05504657453517e-03	1.16387592434738e+68
0.875	1.09095673467859e+68	2.62914973706271e-03	-1.09095673467859e+68
0.9	-9.65369245464861e+67	2.16916491087679e-03	9.65369245464861e+67
0.925	7.87486147424057e+67	1.67557655440888e-03	-7.87486147424057e+67
0.95	-5.61463676441038e+67	1.14899388963470e-03	5.61463676441038e+67
0.975	2.95215226265739e+67	5.90166343093643e-04	-2.95215226265739e+67
1	0.00000000000000e+00	9.44092120664098e-19	9.44092120664098e-19

K=40 N=1000 T=0.1 A=2 nu=1
dx=0.025 dt=0.0001 R=0.004 rho=0.16

#	xk	uk	vk	vk-uk
0	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00
0.025	3.05976854556274e-04	3.05821339524153e-04	-1.55515032120685e-07	-1.55515032120685e-07
0.05	6.24590550263002e-04	6.24271147151420e-04	-3.19403111581911e-07	-3.19403111581911e-07
0.075	9.53329451366152e-04	9.52840265494388e-04	-4.89185871763295e-07	-4.89185871763295e-07
0.1	1.28949468678490e-03	1.28883246373243e-03	-6.62223052474509e-07	-6.62223052474509e-07
0.125	1.63023961926961e-03	1.62940382814589e-03	-8.35791123713443e-07	-8.35791123713443e-07
0.15	1.97261207836128e-03	1.97160491314917e-03	-1.00716521210823e-06	-1.00716521210823e-06
0.175	2.31359809251378e-03	2.31242439169945e-03	-1.17370081433379e-06	-1.17370081433379e-06
0.2	2.65016580674095e-03	2.64883289500422e-03	-1.33291173672187e-06	-1.33291173672187e-06
0.225	2.97930827769127e-03	2.97782573685322e-03	-1.48254083805286e-06	-1.48254083805286e-06
0.25	3.29808389836261e-03	3.29646327789337e-03	-1.62062046924014e-06	-1.62062046924014e-06
0.275	3.60365331724405e-03	3.60190779725900e-03	-1.74551998504248e-06	-1.74551998504248e-06
0.3	3.89331187636794e-03	3.89145589804292e-03	-1.85597832502325e-06	-1.85597832502325e-06
0.325	4.16451679202904e-03	4.16256567163794e-03	-1.95112039110077e-06	-1.95112039110077e-06
0.35	4.41490853121623e-03	4.41287807446811e-03	-2.03045674812118e-06	-2.03045674812118e-06
0.375	4.64232608502404e-03	4.64023221802582e-03	-2.09386699821616e-06	-2.09386699821616e-06
0.4	4.84481609545037e-03	4.84267452746652e-03	-2.14156798385764e-06	-2.14156798385764e-06
0.425	5.02063604174514e-03	5.01846197303120e-03	-2.17406871394401e-06	-2.17406871394401e-06
0.45	5.16825192493691e-03	5.16605981039609e-03	-2.19211454081675e-06	-2.19211454081675e-06
0.475	5.28633109344821e-03	5.28413446983959e-03	-2.19662360861992e-06	-2.19662360861992e-06
0.5	5.37373101955998e-03	5.37154240064203e-03	-2.18861891795093e-06	-2.18861891795093e-06
0.525	5.42948495877834e-03	5.42731579928773e-03	-2.16915949061060e-06	-2.16915949061060e-06
0.55	5.45278549728304e-03	5.45064622321818e-03	-2.13927406485931e-06	-2.13927406485931e-06
0.575	5.44296701474561e-03	5.44086711423556e-03	-2.09990051005821e-06	-2.09990051005821e-06
0.6	5.39948806188598e-03	5.39743622815100e-03	-2.05183373497720e-06	-2.05183373497720e-06
0.625	5.32191457794292e-03	5.31991889364165e-03	-1.99568430127219e-06	-1.99568430127219e-06
0.65	5.20990475905006e-03	5.20797290977389e-03	-1.93184927617471e-06	-1.93184927617471e-06
0.675	5.06319624276237e-03	5.06133574665621e-03	-1.86049610616246e-06	-1.86049610616246e-06
0.7	4.88159610673994e-03	4.87981454722936e-03	-1.78155951057607e-06	-1.78155951057607e-06
0.725	4.66497400196994e-03	4.66327925134288e-03	-1.69475062705964e-06	-1.69475062705964e-06
0.75	4.41325856436960e-03	4.41165898743486e-03	-1.59957693474187e-06	-1.59957693474187e-06

0.775	4.12643708430974e-03	4.12494171343100e-03	-1.49537087874678e-06
0.8	3.80455827166491e-03	3.80317694700718e-03	-1.38132465773034e-06
0.825	3.44773784292131e-03	3.44648131457678e-03	-1.25652834452464e-06
0.85	3.05616658294396e-03	3.05504657453517e-03	-1.12000840879175e-06
0.875	2.63012050087300e-03	2.62914973706271e-03	-9.70763810293103e-07
0.9	2.16997270800790e-03	2.16916491087679e-03	-8.07797131108321e-07
0.925	1.67620669311132e-03	1.67557655440888e-03	-6.30138702440537e-07
0.95	1.14943075196863e-03	1.14899388963470e-03	-4.36862333930100e-07
0.975	5.90393435129974e-04	5.90166343093643e-04	-2.27092036331460e-07
1	0.00000000000000e+00	9.44092120664098e-19	9.44092120664098e-19

K=40 N=900 T=0.9 A=2 nu=1
dx=0.025 dt=0.001 R=0.04 rho=1.6

#	xk	uk	vk	vk-uk
	0	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00
0.025		-nan	4.40574551492635e-08	-nan
0.05		-nan	9.00670446937724e-08	-nan
0.075		-nan	1.37808525795432e-07	-nan
0.1		-nan	1.87038305638134e-07	-nan
0.125		-nan	2.37489940206277e-07	-nan
0.15		-nan	2.88874817732295e-07	-nan
0.175		-nan	3.40883031523022e-07	-nan
0.2		-nan	3.93184445797730e-07	-nan
0.225		-nan	4.45429957013086e-07	-nan
0.25		-nan	4.97252951916199e-07	-nan
0.275		-nan	5.48270962260868e-07	-nan
0.3		-nan	5.98087514749589e-07	-nan
0.325		-nan	6.46294173330988e-07	-nan
0.35		-nan	6.92472769495930e-07	-nan
0.375		-nan	7.36197814683292e-07	-nan
0.4		-nan	7.77039087336565e-07	-nan
0.425		-nan	8.14564385554046e-07	-nan
0.45		-nan	8.48342434658105e-07	-nan
0.475		-nan	8.77945937383140e-07	-nan
0.5		-nan	9.02954752758228e-07	-nan
0.525		-nan	9.22959188150546e-07	-nan
0.55		-nan	9.37563387351391e-07	-nan
0.575		-nan	9.46388796040246e-07	-nan
0.6		-nan	9.49077684466671e-07	-nan
0.625		-nan	9.45296705757803e-07	-nan
0.65		-nan	9.34740466904159e-07	-nan
0.675		-nan	9.17135088211605e-07	-nan
0.7		-nan	8.92241725846328e-07	-nan
0.725		-nan	8.59860031055625e-07	-nan
0.75		-nan	8.19831518733828e-07	-nan
0.775		-nan	7.72042817232526e-07	-nan
0.8		-nan	7.16428770700332e-07	-nan
0.825		-nan	6.52975364791892e-07	-nan
0.85		-nan	5.81722446320546e-07	-nan
0.875		-nan	5.02766207354951e-07	-nan
0.9		-nan	4.16261404387515e-07	-nan
0.925		-nan	3.22423283541064e-07	-nan
0.95		-nan	2.21529183338179e-07	-nan
0.975		-nan	1.13919787342593e-07	-nan

1 0.000000000000000e+00 1.82309683727663e-22 1.82309683727663e-22

K=40 N=9000 T=0.9 A=2 nu=1
dx=0.025 dt=0.0001 R=0.004 rho=0.16

#	xk	uk	vk	vk-uk
0	0.000000000000000e+00	0.000000000000000e+00	0.000000000000000e+00	0.000000000000000e+00
0.025	4.41457150206448e-08	4.40574551492635e-08	-8.82598713813266e-11	-8.82598713813266e-11
0.05	9.02479453448831e-08	9.00670446937724e-08	-1.80900651110668e-10	-1.80900651110668e-10
0.075	1.38086035203123e-07	1.37808525795432e-07	-2.77509407691082e-10	-2.77509407691082e-10
0.1	1.87415927106211e-07	1.87038305638134e-07	-3.77621468076951e-10	-3.77621468076951e-10
0.125	2.37970661023975e-07	2.37489940206277e-07	-4.80720817698138e-10	-4.80720817698138e-10
0.15	2.89461058623160e-07	2.88874817732295e-07	-5.86240890865102e-10	-5.86240890865102e-10
0.175	3.41576597290909e-07	3.40883031523022e-07	-6.93565767886827e-10	-6.93565767886827e-10
0.2	3.93986477590964e-07	3.93184445797730e-07	-8.02031793233560e-10	-8.02031793233560e-10
0.225	4.46340886639947e-07	4.45429957013086e-07	-9.10929626860611e-10	-9.10929626860611e-10
0.25	4.98272458654583e-07	4.97252951916199e-07	-1.01950673838350e-09	-1.01950673838350e-09
0.275	5.49397932612028e-07	5.48270962260868e-07	-1.12697035116057e-09	-1.12697035116057e-09
0.3	5.99320005590099e-07	5.98087514749589e-07	-1.23249084051005e-09	-1.23249084051005e-09
0.325	6.47629378918256e-07	6.46294173330988e-07	-1.33520558726796e-09	-1.33520558726796e-09
0.35	6.93906992780622e-07	6.92472769495930e-07	-1.43422328469238e-09	-1.43422328469238e-09
0.375	7.37726443376653e-07	7.36197814683292e-07	-1.52862869336094e-09	-1.52862869336094e-09
0.4	7.78656575171757e-07	7.77039087336565e-07	-1.61748783519114e-09	-1.61748783519114e-09
0.425	8.16264239168112e-07	8.14564385554046e-07	-1.69985361406566e-09	-1.69985361406566e-09
0.45	8.50117206504892e-07	8.48342434658105e-07	-1.77477184678772e-09	-1.77477184678772e-09
0.475	8.79787225067370e-07	8.77945937383140e-07	-1.84128768422949e-09	-1.84128768422949e-09
0.5	9.04853205156844e-07	9.02954752758228e-07	-1.89845239861586e-09	-1.89845239861586e-09
0.525	9.24904518659452e-07	9.22959188150546e-07	-1.94533050890612e-09	-1.94533050890612e-09
0.55	9.39544394563633e-07	9.37563387351391e-07	-1.98100721224176e-09	-1.98100721224176e-09
0.575	9.48393392125685e-07	9.46388796040246e-07	-2.00459608543830e-09	-2.00459608543830e-09
0.6	9.51092931483216e-07	9.49077684466671e-07	-2.01524701654486e-09	-2.01524701654486e-09
0.625	9.47308860080405e-07	9.45296705757803e-07	-2.01215432260158e-09	-2.01215432260158e-09
0.65	9.36735031910093e-07	9.34740466904159e-07	-1.99456500593486e-09	-1.99456500593486e-09
0.675	9.19096875309270e-07	9.17135088211605e-07	-1.96178709766504e-09	-1.96178709766504e-09
0.7	8.94154923879929e-07	8.92241725846328e-07	-1.91319803360016e-09	-1.91319803360016e-09
0.725	8.61708284060011e-07	8.59860031055625e-07	-1.84825300438577e-09	-1.84825300438577e-09
0.75	8.21598011952535e-07	8.19831518733828e-07	-1.76649321870690e-09	-1.76649321870690e-09
0.775	7.73710371248068e-07	7.72042817232526e-07	-1.66755401554238e-09	-1.66755401554238e-09
0.8	7.17979943459278e-07	7.16428770700332e-07	-1.55117275894684e-09	-1.55117275894684e-09
0.825	6.54392561238593e-07	6.52975364791892e-07	-1.41719644670105e-09	-1.41719644670105e-09
0.85	5.82988035282889e-07	5.81722446320546e-07	-1.26558896234271e-09	-1.26558896234271e-09
0.875	5.03862645253675e-07	5.02766207354951e-07	-1.09643789872384e-09	-1.09643789872384e-09
0.9	4.17171365267794e-07	4.16261404387515e-07	-9.09960880278871e-10	-9.09960880278871e-10
0.925	3.23129794851754e-07	3.22423283541064e-07	-7.06511310689892e-10	-7.06511310689892e-10
0.95	2.22015766810853e-07	2.21529183338179e-07	-4.86583472673758e-10	-4.86583472673758e-10
0.975	1.14170604249725e-07	1.13919787342593e-07	-2.50816907132067e-10	-2.50816907132067e-10
1	0.000000000000000e+00	1.82309683727663e-22	1.82309683727663e-22	1.82309683727663e-22

K=40 N=60 T=0.06 A=2 nu=0.01
dx=0.025 dt=0.001 R=0.04 rho=0.016

#	xk	uk	vk	vk-uk
0	0.000000000000000e+00	0.000000000000000e+00	0.000000000000000e+00	0.000000000000000e+00
0.025	-2.16569580025298e-03	6.74482781277200e-04	2.84017858153018e-03	2.84017858153018e-03
0.05	-1.50719828677355e-02	1.30037164365576e-03	1.63723545113912e-02	1.63723545113912e-02

0.075	-7.85988037906078e-03	1.80523803757166e-03	9.66511841663244e-03
0.1	6.45456916935109e-02	2.13488161551833e-03	-6.24108100779925e-02
0.125	2.15900801916273e-01	2.25577597283107e-03	-2.13645025943442e-01
0.15	4.19399368045152e-01	2.15588150061455e-03	-4.17243486544538e-01
0.175	6.28924891058794e-01	1.84495569249225e-03	-6.27079935366302e-01
0.2	8.01060918865144e-01	1.35343281898829e-03	-7.99707486046156e-01
0.225	9.05099224920890e-01	7.29426512483536e-04	-9.04369798408407e-01
0.25	9.24294456246690e-01	3.40188567416987e-05	-9.24260437389948e-01
0.275	8.54200318621397e-01	-6.64718801721162e-04	-8.54865037423118e-01
0.3	7.00821660253234e-01	-1.29838915250333e-03	-7.02120049405737e-01
0.325	4.78923645752567e-01	-1.80496412662929e-03	-4.80728609879196e-01
0.35	2.10163489935116e-01	-2.13485663611623e-03	-2.12298346571233e-01
0.375	-7.91652613404142e-02	-2.25577450363929e-03	7.69094868367749e-02
0.4	-3.60744093429384e-01	-2.15588144584100e-03	3.58588211983543e-01
0.425	-6.07010667871775e-01	-1.84495569121451e-03	6.05165712180561e-01
0.45	-7.93858791621192e-01	-1.35343281896978e-03	7.92505358802222e-01
0.475	-9.02998483274516e-01	-7.29426512483346e-04	9.02269056762033e-01
0.5	-9.23746391507410e-01	-3.40188567416988e-05	9.23712372650668e-01
0.525	-8.54071566781730e-01	6.64718801721132e-04	8.54736285583451e-01
0.55	-7.00794266428659e-01	1.29838915250327e-03	7.02092655581163e-01
0.575	-4.78918340556343e-01	1.80496412662922e-03	4.80723304682973e-01
0.6	-2.10162550690008e-01	2.13485663611614e-03	2.12297407326124e-01
0.625	7.91654139259427e-02	2.25577450363920e-03	-7.69096394223035e-02
0.65	3.60744116256481e-01	2.15588144584090e-03	-3.58588234810640e-01
0.675	6.07010670935424e-01	1.84495569121442e-03	-6.05165715244210e-01
0.7	7.93858793075645e-01	1.35343281896972e-03	-7.92505360256676e-01
0.725	9.02998471787475e-01	7.29426512483309e-04	-9.02269045274992e-01
0.75	9.23746506080844e-01	3.40188567416989e-05	-9.23712487224102e-01
0.775	8.54070542118058e-01	-6.64718801720993e-04	-8.54735260919779e-01
0.8	7.00802463269192e-01	-1.29838915249082e-03	-7.02100852421683e-01
0.825	4.78860145174661e-01	-1.80496412572876e-03	-4.80665109300390e-01
0.85	2.10526001960841e-01	-2.13485659553884e-03	-2.12660858556380e-01
0.875	-8.11423699138815e-02	-2.25577335943143e-03	7.88865965544500e-02
0.9	-3.51476574545122e-01	-2.15586099443615e-03	3.49320713550686e-01
0.925	-6.44091074593528e-01	-1.84471993388145e-03	6.42246354659647e-01
0.95	-6.67909801165711e-01	-1.35163898679403e-03	6.66558162178917e-01
0.975	-1.26810099945739e+00	-7.20138727829358e-04	1.26738086072956e+00
1	0.00000000000000e+00	-1.14739460755934e-18	-1.14739460755934e-18

K=40 N=600 T=0.06 A=2 nu=0.01

dx=0.025 dt=0.0001 R=0.004 rho=0.0016

#	xk	uk	vk	vk-uk
0	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00
0.025	-2.76735708309760e-03	6.74482781277200e-04	3.44183986437480e-03	3.44183986437480e-03
0.05	-1.30478218829001e-02	1.30037164365576e-03	1.43481935265559e-02	1.43481935265559e-02
0.075	1.41859209161105e-04	1.80523803757166e-03	1.66337882841055e-03	1.66337882841055e-03
0.1	7.63954479361775e-02	2.13488161551833e-03	-7.42605663206591e-02	-7.42605663206591e-02
0.125	2.25856801736794e-01	2.25577597283107e-03	-2.23601025763963e-01	-2.23601025763963e-01
0.15	4.23066987373473e-01	2.15588150061455e-03	-4.20911105872858e-01	-4.20911105872858e-01
0.175	6.25282987900664e-01	1.84495569249225e-03	-6.23438032208172e-01	-6.23438032208172e-01
0.2	7.91482476779465e-01	1.35343281898829e-03	-7.90129043960477e-01	-7.90129043960477e-01
0.225	8.91797039054137e-01	7.29426512483536e-04	-8.91067612541654e-01	-8.91067612541654e-01
0.25	9.09453674552235e-01	3.40188567416987e-05	-9.09419655695493e-01	-9.09419655695493e-01
0.275	8.39729524385699e-01	-6.64718801721162e-04	-8.40394243187421e-01	-8.40394243187421e-01

0.3	6.88329317940860e-01	-1.29838915250333e-03	-6.89627707093363e-01
0.325	4.69701467999588e-01	-1.80496412662929e-03	-4.71506432126217e-01
0.35	2.05135825078686e-01	-2.13485663611623e-03	-2.07270681714803e-01
0.375	-7.95002274554946e-02	-2.25577450363929e-03	7.72444529518553e-02
0.4	-3.56352047205362e-01	-2.15588144584100e-03	3.54196165759521e-01
0.425	-5.98321182466988e-01	-1.84495569121451e-03	5.96476226775773e-01
0.45	-7.81722380109942e-01	-1.35343281896978e-03	7.80368947290973e-01
0.475	-8.88603127598654e-01	-7.29426512483346e-04	8.87873701086171e-01
0.5	-9.08501206712875e-01	-3.40188567416988e-05	9.08467187856133e-01
0.525	-8.39468857318638e-01	6.64718801721132e-04	8.40133576120359e-01
0.55	-6.88263447168534e-01	1.29838915250327e-03	6.89561836321038e-01
0.575	-4.69686015384469e-01	1.80496412662922e-03	4.71490979511099e-01
0.6	-2.05132443917802e-01	2.13485663611614e-03	2.07267300553918e-01
0.625	7.95009203991104e-02	2.25577450363920e-03	-7.72451458954712e-02
0.65	3.56352180739905e-01	2.15588144584090e-03	-3.54196299294064e-01
0.675	5.98321206476518e-01	1.84495569121442e-03	-5.96476250785304e-01
0.7	7.81722386810874e-01	1.35343281896972e-03	-7.80368953991904e-01
0.725	8.88603104926141e-01	7.29426512483309e-04	-8.87873678413657e-01
0.75	9.08501405923523e-01	3.40188567416989e-05	-9.08467387066781e-01
0.775	8.39467299462450e-01	-6.64718801720993e-04	-8.40132018264171e-01
0.8	6.88274555267422e-01	-1.29838915249082e-03	-6.89572944419912e-01
0.825	4.69614415704568e-01	-1.80496412572876e-03	-4.71419379830297e-01
0.85	2.05545835396506e-01	-2.13485659553884e-03	-2.07680691992045e-01
0.875	-8.16169719966392e-02	-2.25577335943143e-03	7.93611986372078e-02
0.9	-3.46854571703070e-01	-2.15586099443615e-03	3.44698710708634e-01
0.925	-6.35304742048683e-01	-1.84471993388145e-03	6.33460022114802e-01
0.95	-6.57703187800303e-01	-1.35163898679403e-03	6.56351548813509e-01
0.975	-1.24718948552957e+00	-7.20138727829358e-04	1.24646934680174e+00
1	0.00000000000000e+00	-1.14739460755934e-18	-1.14739460755934e-18

K=40 N=100 T=0.1 A=2 nu=0.01
dx=0.025 dt=0.001 R=0.04 rho=0.016

#	xk	uk	vk	vk-uk
0	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00
0.025	3.60751546065551e-04	1.14507808794966e-05	-3.49300765186055e-04	-3.49300765186055e-04
0.05	2.27596215346268e-03	2.21408302683046e-05	-2.25382132319437e-03	-2.25382132319437e-03
0.075	-2.19847835154666e-04	3.08452524794874e-05	2.50693087634153e-04	2.50693087634153e-04
0.1	-1.15896308264578e-02	3.66012401000233e-05	1.16262320665578e-02	1.16262320665578e-02
0.125	-1.96346959010294e-02	3.87956697613830e-05	1.96734915707908e-02	1.96734915707908e-02
0.15	3.46476573449652e-03	3.71973461843038e-05	-3.42756838831221e-03	-3.42756838831221e-03
0.175	8.26100545044820e-02	3.19587216849241e-05	-8.25780957827971e-02	-8.25780957827971e-02
0.2	2.22249038170805e-01	2.35918633298297e-05	-2.22225446307475e-01	-2.22225446307475e-01
0.225	4.03226281327381e-01	1.29156796091483e-05	-4.03213365647771e-01	-4.03213365647771e-01
0.25	5.91956900459210e-01	9.75219953555280e-07	-5.91955925239257e-01	-5.91955925239257e-01
0.275	7.52663785062320e-01	-1.10607009836022e-05	-7.52674845763303e-01	-7.52674845763303e-01
0.3	8.56153624979782e-01	-2.20139234423312e-05	-8.56175638903224e-01	-8.56175638903224e-01
0.325	8.83837058854760e-01	-3.08122696944493e-05	-8.83867871124455e-01	-8.83867871124455e-01
0.35	8.28485258402670e-01	-3.65944963071668e-05	-8.28521852898977e-01	-8.28521852898977e-01
0.375	6.93405192908746e-01	-3.87945986524906e-05	-6.93443987507398e-01	-6.93443987507398e-01
0.4	4.90941080056996e-01	-3.71972153838459e-05	-4.90978277272380e-01	-4.90978277272380e-01
0.425	2.40581940147234e-01	-3.19587095051914e-05	-2.40613898856739e-01	-2.40613898856739e-01
0.45	-3.32779254423097e-02	-2.35918624707464e-05	3.32543335798390e-02	3.32543335798390e-02
0.475	-3.03866489601200e-01	-1.29156795634936e-05	3.03853573921637e-01	3.03853573921637e-01
0.5	-5.44706861841828e-01	-9.75219951735159e-07	5.44705886621876e-01	5.44705886621876e-01

0.525	-7.32226644737041e-01	1.10607009836560e-05	7.32237705438025e-01
0.55	-8.48070778705660e-01	2.20139234423315e-05	8.48092792629103e-01
0.575	-8.80899792260996e-01	3.08122696944480e-05	8.80930604530690e-01
0.6	-8.27500187549985e-01	3.65944963071652e-05	8.27536782046292e-01
0.625	-6.93099098408899e-01	3.87945986524889e-05	6.93137893007551e-01
0.65	-4.90852631282371e-01	3.71972153838442e-05	4.90889828497755e-01
0.675	-2.40558151054139e-01	3.19587095051899e-05	2.40590109763644e-01
0.7	3.32842463001241e-02	2.35918624707460e-05	-3.32606544376534e-02
0.725	3.03866080142089e-01	1.29156795635242e-05	-3.03853164462526e-01
0.75	5.44717055042780e-01	9.75219952839121e-07	-5.44716079822827e-01
0.775	7.32178247598187e-01	-1.10607009545449e-05	-7.32189308299142e-01
0.8	8.48285855603224e-01	-2.20139228664701e-05	-8.48307869526090e-01
0.825	8.80041589647625e-01	-3.08122611115345e-05	-8.80072401908737e-01
0.85	8.30563175352486e-01	-3.65943994078017e-05	-8.30599769751894e-01
0.875	6.83361973167794e-01	-3.87937644720436e-05	-6.83400766932266e-01
0.9	5.18378150585345e-01	-3.71916940332391e-05	-5.18415342279378e-01
0.925	1.71233442182747e-01	-3.19303209590219e-05	-1.71265372503706e-01
0.95	1.22902202524302e-01	-2.34770324260008e-05	-1.22925679556728e-01
0.975	-6.17833966434710e-01	-1.25446240886633e-05	6.17821421810621e-01
1	0.00000000000000e+00	-2.01080853901805e-20	-2.01080853901805e-20

K=40 N=1000 T=0.1 A=2 nu=0.01
dx=0.025 dt=0.0001 R=0.004 rho=0.0016

#	xk	uk	vk	vk-uk
0	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00
0.025	4.57483064188449e-04	1.14507808794966e-05	-4.46032283308952e-04	-4.46032283308952e-04
0.05	1.65736351688063e-03	2.21408302683046e-05	-1.63522268661233e-03	-1.63522268661233e-03
0.075	-1.61831889870745e-03	3.08452524794874e-05	1.64916415118694e-03	1.64916415118694e-03
0.1	-1.10076517826551e-02	3.66012401000233e-05	1.10442530227551e-02	1.10442530227551e-02
0.125	-1.33155133096975e-02	3.87956697613830e-05	1.33543089794589e-02	1.33543089794589e-02
0.15	1.62870053830123e-02	3.71973461843038e-05	-1.62498080368280e-02	-1.62498080368280e-02
0.175	9.82934713826226e-02	3.19587216849241e-05	-9.82615126609377e-02	-9.82615126609377e-02
0.2	2.34998845295275e-01	2.35918633298297e-05	-2.34975253431945e-01	-2.34975253431945e-01
0.225	4.08348186294659e-01	1.29156796091483e-05	-4.08335270615050e-01	-4.08335270615050e-01
0.25	5.87584810723574e-01	9.75219953555280e-07	-5.87583835503620e-01	-5.87583835503620e-01
0.275	7.39606314445531e-01	-1.10607009836022e-05	-7.39617375146514e-01	-7.39617375146514e-01
0.3	8.36846572786573e-01	-2.20139234423312e-05	-8.36868586710015e-01	-8.36868586710015e-01
0.325	8.61333488438590e-01	-3.08122696944493e-05	-8.61364300708284e-01	-8.61364300708284e-01
0.35	8.05824951262833e-01	-3.65944963071668e-05	-8.05861545759140e-01	-8.05861545759140e-01
0.375	6.73298199285116e-01	-3.87945986524906e-05	-6.73336993883768e-01	-6.73336993883768e-01
0.4	4.75604420578336e-01	-3.71972153838459e-05	-4.75641617793719e-01	-4.75641617793719e-01
0.425	2.31628278488035e-01	-3.19587095051914e-05	-2.31660237197540e-01	-2.31660237197540e-01
0.45	-3.49271908967726e-02	-2.35918624707464e-05	3.49035990343019e-02	3.49035990343019e-02
0.475	-2.98033366072985e-01	-1.29156795634936e-05	2.98020450393422e-01	2.98020450393422e-01
0.5	-5.31956722486831e-01	-9.75219951735159e-07	5.31955747266880e-01	5.31955747266880e-01
0.525	-7.13805795127301e-01	1.10607009836560e-05	7.13816855828284e-01	7.13816855828284e-01
0.55	-8.25781860688862e-01	2.20139234423315e-05	8.25803874612305e-01	8.25803874612305e-01
0.575	-8.56924457245383e-01	3.08122696944480e-05	8.56955269515078e-01	8.56955269515078e-01
0.6	-8.04185270977713e-01	3.65944963071652e-05	8.04221865474020e-01	8.04221865474020e-01
0.625	-6.72726818665235e-01	3.87945986524889e-05	6.72765613263887e-01	6.72765613263887e-01
0.65	-4.75417160642466e-01	3.71972153838442e-05	4.75454357857850e-01	4.75454357857850e-01
0.675	-2.31570454254425e-01	3.19587095051899e-05	2.31602412963930e-01	2.31602412963930e-01
0.7	3.49445561458448e-02	2.35918624707460e-05	-3.49209642833740e-02	-3.49209642833740e-02
0.725	2.98035757894370e-01	1.29156795635242e-05	-2.98022842214807e-01	-2.98022842214807e-01

0.75	5.31969592846141e-01	9.75219952839121e-07	-5.31968617626188e-01
0.775	7.13752302619943e-01	-1.10607009545449e-05	-7.13763363320897e-01
0.8	8.26009620942493e-01	-2.20139228664701e-05	-8.26031634865359e-01
0.825	8.56048488392254e-01	-3.08122611115345e-05	-8.56079300653366e-01
0.85	8.07232069705842e-01	-3.65943994078017e-05	-8.07268664105250e-01
0.875	6.63192876712722e-01	-3.87937644720436e-05	-6.63231670477194e-01
0.9	5.02175767741027e-01	-3.71916940332391e-05	-5.02212959435060e-01
0.925	1.64243075024821e-01	-3.19303209590219e-05	-1.64275005345780e-01
0.95	1.17034515521942e-01	-2.34770324260008e-05	-1.17057992554368e-01
0.975	-6.04151036834756e-01	-1.25446240886633e-05	6.04138492210668e-01
1	0.00000000000000e+00	-2.01080853901805e-20	-2.01080853901805e-20

K=40 N=900 T=0.9 A=2 nu=0.01
dx=0.025 dt=0.001 R=0.04 rho=0.016

#	xk	uk	vk	vk-uk
0	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00
0.025	1.84985662554915e-15	5.69176277006497e-41	-1.84985662554915e-15	-1.84985662554915e-15
0.05	-7.90716713626364e-15	1.11340650376665e-40	7.90716713626364e-15	7.90716713626364e-15
0.075	2.87615393814742e-15	1.58017751053719e-40	-2.87615393814742e-15	-2.87615393814742e-15
0.1	-5.56174322549976e-15	1.92263668825717e-40	5.56174322549976e-15	5.56174322549976e-15
0.125	8.72665830375415e-14	2.10464432075706e-40	-8.72665830375415e-14	-8.72665830375415e-14
0.15	-1.97601518388646e-13	2.10483169779242e-40	1.97601518388646e-13	1.97601518388646e-13
0.175	1.87593333226661e-13	1.91919224298881e-40	-1.87593333226661e-13	-1.87593333226661e-13
0.2	-4.70767977513962e-13	1.56189542823066e-40	4.70767977513962e-13	4.70767977513962e-13
0.225	1.59445669413877e-12	1.06421154639127e-40	-1.59445669413877e-12	-1.59445669413877e-12
0.25	-2.50190115111478e-12	4.71644314004358e-41	2.50190115111478e-12	2.50190115111478e-12
0.275	2.72491078449753e-12	-1.60437102664431e-41	-2.72491078449753e-12	-2.72491078449753e-12
0.3	-6.30934969206897e-12	-7.72216104333435e-41	6.30934969206897e-12	6.30934969206897e-12
0.325	1.27304097943712e-11	-1.30533871696575e-40	-1.27304097943712e-11	-1.27304097943712e-11
0.35	-1.04251829002714e-11	-1.70871084303654e-40	1.04251829002714e-11	1.04251829002714e-11
0.375	5.29984151708115e-12	-1.94359402177743e-40	-5.29984151708115e-12	-5.29984151708115e-12
0.4	-1.70621253080506e-11	-1.98748635243617e-40	1.70621253080506e-11	1.70621253080506e-11
0.425	1.13526502639279e-12	-1.83640058601396e-40	-1.13526502639279e-12	-1.13526502639279e-12
0.45	8.09360288905410e-11	-1.50531340252608e-40	-8.09360288905410e-11	-8.09360288905410e-11
0.475	-9.92920503554501e-11	-1.02674229618592e-40	9.92920503554501e-11	9.92920503554501e-11
0.5	6.33927560215971e-11	-4.47591518289244e-41	-6.33927560215971e-11	-6.33927560215971e-11
0.525	-2.47379696920964e-10	1.75421591516285e-41	2.47379696920964e-10	2.47379696920964e-10
0.55	2.18741042264656e-10	7.81310041861009e-41	-2.18741042264656e-10	-2.18741042264656e-10
0.575	5.58813447742500e-10	1.31078661081580e-40	-5.58813447742500e-10	-5.58813447742500e-10
0.6	-7.08522570454663e-10	1.71207503603697e-40	7.08522570454663e-10	7.08522570454663e-10
0.625	3.73907482914940e-10	1.94599559343500e-40	-3.73907482914940e-10	-3.73907482914940e-10
0.65	-2.23088994590210e-09	1.98982993595506e-40	2.23088994590210e-09	2.23088994590210e-09
0.675	1.10377126427539e-09	1.83959037602844e-40	-1.10377126427539e-09	-1.10377126427539e-09
0.7	6.12480707887974e-09	1.51047496818956e-40	-6.12480707887974e-09	-6.12480707887974e-09
0.725	-4.41536004699204e-09	1.03546593743996e-40	4.41536004699204e-09	4.41536004699204e-09
0.75	2.53284247981476e-09	4.62207356998431e-41	-2.53284247981476e-09	-2.53284247981476e-09
0.775	-1.52649394448943e-08	-1.51539394551704e-41	1.52649394448943e-08	1.52649394448943e-08
0.8	-1.70531317052818e-08	-7.43432877196221e-41	1.70531317052818e-08	1.70531317052818e-08
0.825	5.89944605206133e-08	-1.25256468927404e-40	-5.89944605206133e-08	-5.89944605206133e-08
0.85	4.39923795754995e-09	-1.62539465301124e-40	-4.39923795754995e-09	-4.39923795754995e-09
0.875	1.01802768093377e-08	-1.82106858894987e-40	-1.01802768093377e-08	-1.01802768093377e-08
0.9	3.23728802457661e-09	-1.81563981832394e-40	-3.23728802457661e-09	-3.23728802457661e-09
0.925	-3.50076814542272e-07	-1.60481333543321e-40	3.50076814542272e-07	3.50076814542272e-07
0.95	6.51048631759363e-08	-1.20495500508405e-40	-6.51048631759363e-08	-6.51048631759363e-08

```

0.975    4.84945305516637e-07    -6.52245490673213e-41    -4.84945305516637e-07
      1    0.00000000000000e+00    -1.06147935096202e-55    -1.06147935096202e-55

```

```

# K=40 N=9000 T=0.9 A=2 nu=0.01
# dx=0.025 dt=0.0001 R=0.004 rho=0.0016

```

#	xk	uk	vk	vk-uk
0	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00
0.025	6.37880293192940e-16	5.69176277006497e-41	-6.37880293192940e-16	-6.37880293192940e-16
0.05	-3.05180547784293e-15	1.11340650376665e-40	3.05180547784293e-15	3.05180547784293e-15
0.075	9.53159648881081e-15	1.58017751053719e-40	-9.53159648881081e-15	-9.53159648881081e-15
0.1	-2.06172359349070e-14	1.92263668825717e-40	2.06172359349070e-14	2.06172359349070e-14
0.125	4.49155417022674e-14	2.10464432075706e-40	-4.49155417022674e-14	-4.49155417022674e-14
0.15	-1.27170874497623e-13	2.10483169779242e-40	1.27170874497623e-13	1.27170874497623e-13
0.175	2.91223949643237e-13	1.91919224298881e-40	-2.91223949643237e-13	-2.91223949643237e-13
0.2	-5.54334424933615e-13	1.56189542823066e-40	5.54334424933615e-13	5.54334424933615e-13
0.225	1.12015865028539e-12	1.06421154639127e-40	-1.12015865028539e-12	-1.12015865028539e-12
0.25	-2.36793527702918e-12	4.71644314004358e-41	2.36793527702918e-12	2.36793527702918e-12
0.275	4.31513048497107e-12	-1.60437102664431e-41	-4.31513048497107e-12	-4.31513048497107e-12
0.3	-7.06927511202427e-12	-7.72216104333435e-41	7.06927511202427e-12	7.06927511202427e-12
0.325	1.20819515777097e-11	-1.30533871696575e-40	-1.20819515777097e-11	-1.20819515777097e-11
0.35	-1.96459457825331e-11	-1.70871084303654e-40	1.96459457825331e-11	1.96459457825331e-11
0.375	2.58956128671561e-11	-1.94359402177743e-40	-2.58956128671561e-11	-2.58956128671561e-11
0.4	-3.00870750098630e-11	-1.98748635243617e-40	3.00870750098630e-11	3.00870750098630e-11
0.425	3.44939206100727e-11	-1.83640058601396e-40	-3.44939206100727e-11	-3.44939206100727e-11
0.45	-2.01730735777953e-11	-1.50531340252608e-40	2.01730735777953e-11	2.01730735777953e-11
0.475	-3.43035490997512e-11	-1.02674229618592e-40	3.43035490997512e-11	3.43035490997512e-11
0.5	9.70722347858007e-11	-4.47591518289244e-41	-9.70722347858007e-11	-9.70722347858007e-11
0.525	-1.67629452270044e-10	1.75421591516285e-41	1.67629452270044e-10	1.67629452270044e-10
0.55	3.09242889817446e-10	7.81310041861009e-41	-3.09242889817446e-10	-3.09242889817446e-10
0.575	-3.27416593742614e-10	1.31078661081580e-40	3.27416593742614e-10	3.27416593742614e-10
0.6	-2.43783791425553e-11	1.71207503603697e-40	2.43783791425553e-11	2.43783791425553e-11
0.625	3.68018738664143e-10	1.94599559343500e-40	-3.68018738664143e-10	-3.68018738664143e-10
0.65	-8.36580752430367e-10	1.98982993595506e-40	8.36580752430367e-10	8.36580752430367e-10
0.675	2.11176392167967e-09	1.83959037602844e-40	-2.11176392167967e-09	-2.11176392167967e-09
0.7	-1.75020630951350e-09	1.51047496818956e-40	1.75020630951350e-09	1.75020630951350e-09
0.725	-1.45392903845798e-09	1.03546593743996e-40	-1.45392903845798e-09	-1.45392903845798e-09
0.75	2.57610114619530e-09	4.62207356998431e-41	2.57610114619530e-09	2.57610114619530e-09
0.775	-6.26095637242355e-09	-1.51539394551704e-41	-6.26095637242355e-09	-6.26095637242355e-09
0.8	1.33161978308293e-08	-7.43432877196221e-41	-1.33161978308293e-08	-1.33161978308293e-08
0.825	3.90237834018097e-09	-1.25256468927404e-40	-3.90237834018097e-09	-3.90237834018097e-09
0.85	-2.34615533728966e-08	-1.62539465301124e-40	2.34615533728966e-08	2.34615533728966e-08
0.875	1.27951854272766e-08	-1.82106858894987e-40	-1.27951854272766e-08	-1.27951854272766e-08
0.9	-4.84169976329102e-08	-1.81563981832394e-40	4.84169976329102e-08	4.84169976329102e-08
0.925	6.90213163557153e-09	-1.60481333543321e-40	-6.90213163557153e-09	-6.90213163557153e-09
0.95	1.80478096887369e-07	-1.20495500508405e-40	-1.80478096887369e-07	-1.80478096887369e-07
0.975	-4.13670845659603e-08	-6.52245490673213e-41	4.13670845659603e-08	4.13670845659603e-08
1	0.00000000000000e+00	-1.06147935096202e-55	-1.06147935096202e-55	-1.06147935096202e-55

Note that in order to verify the answers with $K = 40$ one needs also to decrease the Δt time steps.

7. [Thomas Problem 1.5.7] Solve the following initial boundary value problem

$$\begin{cases} v_t = \nu v_{xx} + F(x, t), & \text{for } x \in (0, 1) \text{ and } t > 0 \\ v(x, 0) = f(x), & \text{for } x \in [0, 1] \\ v(0, t) = a(t), & \text{for } t \geq 0 \\ v(1, t) = b(t), & \text{for } t \geq 0 \end{cases}$$

where $\nu = 0.1$, $f(x) = 0$, $a(t) = b(t) = 0$ and $F(x, t) = \sin(2\pi x) \sin(4\pi t)$. Use $K = 10$ and $\Delta t = 0.05$ and find solutions at $t = 0.1$, $t = 0.9$ and $t = 2.0$. Resolve the problem in the above choice of parameters using $K = 20$ and $\Delta t = 0.01$.

First we derive the exact solution. Following the techniques used in Problem 1.2.1 let $X_n(x) = \sin(n\pi x)$ and $\lambda_n = -n^2\pi^2$ and write

$$v(x, t) = \sum_{n \in \mathbf{N}} T_n(t) X_n(x).$$

Plugging this into the differential equations yields

$$\sum_{n \in \mathbf{N}} T'_n(t) X_n(x) = \sum_{n \in \mathbf{N}} \nu \lambda_n T_n(t) X_n(x) + \sin(4\pi t) X_2(x).$$

Therefore

$$\begin{cases} T'_n(t) = \nu \lambda_n T_n(t) & \text{for } n \neq 2 \\ T'_2(t) = \nu \lambda_2 T_2(t) + \sin(4\pi t) & \text{for } n = 2. \end{cases}$$

Solution for $T_n(t)$ when $n \neq 2$ is the same as in Problem 1.2.1. When $n = 2$ we proceed as follows:

$$\frac{d}{dt} (T_2(t) \exp(-\nu \lambda_2 t)) = \sin(4\pi t) \exp(-\nu \lambda_2 t).$$

Therefore

$$\begin{aligned} T_2(t) \exp(-\nu \lambda_2 t) - T_2(0) &= \int_0^t \sin(4\pi s) \exp(-\nu \lambda_2 s) ds \\ &= \frac{1 - (\cos(4\pi t) - \nu \pi \sin(4\pi t)) \exp(\nu 4\pi^2 t)}{4\pi(\nu^2 \pi^2 + 1)}. \end{aligned}$$

It follows that

$$\begin{cases} T_n(t) = T_n(0) \exp(-\nu n^2 \pi^2 t) & \text{for } n \neq 2 \\ T_2(t) = T_2(0) \exp(-\nu 4\pi^2 t) + \frac{\exp(-\nu 4\pi^2 t) - \cos(4\pi t) + \nu \pi \sin(4\pi t)}{4\pi(\nu^2 \pi^2 + 1)} & \text{for } n = 2. \end{cases}$$

Solving for the initial condition

$$\sum_{n \in \mathbf{N}} T_n(0) \sin(n\pi x) = 0.$$

yields than $T_n(0) = 0$ for all n . Therefore the solution is

$$v(x, t) = \frac{\exp(-\nu 4\pi^2 t) - \cos(4\pi t) + \nu\pi \sin(4\pi t)}{4\pi(\nu^2\pi^2 + 1)} \sin(2\pi x).$$

Integrating both sides of the equation from t_n to t_{n+1} yields

$$v(x, t_{n+1}) = v(x, t_n) + \int_{t_n}^{t_{n+1}} (\nu v_{xx}(x, s) + F(x, s)) ds$$

Approximating integral using the left endpoint yields

$$v(x, t_{n+1}) \approx v(x, t_n) + \Delta t (\nu v_{xx}(x, t_n) + F(x, t_n))$$

and the explicit method

$$u_k^{n+1} = u_k^n + R\delta^2 u_k^n + F(x_k, t_n).$$

Alternatively, for better accuracy, we can approximate the second term in the intergral using the midpoint method. This yields an explicit scheme of the form

$$u_k^{n+1} = u_k^n + R\delta^2 u_k^n + F(x_k, t_{n+1/2}).$$

The program is

```

1 /* prog7.c -- Thomas Problem 1.5.7
2    Written May 2013 by Eric Olson */
3
4 #include <stdio.h>
5 #include <math.h>
6 #define NU 0.1
7
8 double dt,dx,R,rho;
9
10 double v(double x,double t){
11     return (exp(-NU*4*M_PI*M_PI*t)
12             -cos(4*M_PI*t)+NU*M_PI*sin(4*M_PI*t))
13             /(4*M_PI*(NU*NU*M_PI*M_PI+1))*sin(2*M_PI*x);
14 }
15
16 double F(double x,double t){
17     return sin(2*M_PI*x)*sin(4*M_PI*t);
18 }
19 double f(double x){
20     return 0;
21 }
22 void doinit(int K,double u[K+1]){
23     int k;

```

```

24     for(k=0;k<=K;k++){
25         double xk=k*dx;
26         u[k]=f(xk);
27     }
28 }
29 void doprint(int K,double u[K+1],double tn){
30     int k;
31     printf("#%4s %23s %23s %23s\n","xk","uk","vk","vk-uk");
32     for(k=0;k<=K;k++){
33         double xk=k*dx;
34         double vk=v(xk,tn);
35         printf("%5g %23.14e %23.14e %23.14e\n",xk,u[k],vk,vk-u[k]);
36     }
37 }
38 void dosolve(int K,double u[K+1],int N,double T){
39     int n;
40     printf("# SCHEME=Left\n");
41     for(n=0;;n++){
42         double tn=n*dt;
43         u[0]=0; u[K]=0;
44         if(n>=N) break;
45         int k;
46         double w[K+1];
47         for(k=1;k<K;k++) {
48             double xk=k*dx;
49             w[k]=rho*(u[k+1]-2*u[k]+u[k-1])+dt*F(xk,tn);
50         }
51         for(k=1;k<K;k++) u[k]+=w[k];
52     }
53 }
54 void dosolvemp(int K,double u[K+1],int N,double T){
55     int n;
56     printf("# SCHEME=Midpoint\n");
57     for(n=0;;n++){
58         double tn=n*dt;
59         u[0]=0; u[K]=0;
60         if(n>=N) break;
61         int k;
62         double w[K+1];
63         for(k=1;k<K;k++) {
64             double xk=k*dx;
65             w[k]=rho*(u[k+1]-2*u[k]+u[k-1])+dt*F(xk,tn+dt/2);
66         }
67         for(k=1;k<K;k++) u[k]+=w[k];
68     }

```

```

69 }
70 main(){
71     printf("# prog7 -- Thomas Problem 1.5.7\n"
72           "# Written May 2013 by Eric Olson\n#\n");
73     int KK[]={10,10,10,20,20,20};
74     double TT[]={0.1,0.9,2.0,0.1,0.9,2.0};
75     int NN[]={2,18,40,10,90,200};
76     int J=sizeof(TT)/sizeof(double);
77     int j;
78     for(j=0;j<J;j++){
79         int K=KK[j],N=NN[j];
80         double T=TT[j],u[K+1];
81         dx=1.0/K; dt=T/N;
82         rho=NU*dt/dx/dx;
83         printf("#\tK=%d N=%d T=%g NU=%g\n"
84               "\tdx=%g dt=%g rho=%g\n",
85               K,N,T,NU,dx,dt,rho);
86         doinit(K,u);
87         dosolve(K,u,N,T);
88         doprint(K,u,T);
89         printf("\n\n");
90         printf("#\tK=%d N=%d T=%g NU=%g\n"
91               "\tdx=%g dt=%g rho=%g\n",
92               K,N,T,NU,dx,dt,rho);
93         doinit(K,u);
94         dosolvemp(K,u,N,T);
95         doprint(K,u,T);
96         printf("\n\n");
97     }
98     return 0;
99 }

```

The output is

```

# prog7 -- Thomas Problem 1.5.7
# Written May 2013 by Eric Olson
#
# K=10 N=2 T=0.1 NU=0.1
# dx=0.1 dt=0.05 rho=0.5
# SCHEME=Left
#   xk                uk                vk                vk-uk
0     0.000000000000000e+00  0.000000000000000e+00  0.000000000000000e+00
0.1   1.72745751406263e-02  2.82508935294770e-02  1.09763183888507e-02
0.2   2.79508497187474e-02  4.57109059432483e-02  1.77600562245009e-02
0.3   2.79508497187474e-02  4.57109059432483e-02  1.77600562245009e-02
0.4   1.72745751406263e-02  2.82508935294770e-02  1.09763183888507e-02
0.5   3.59902777320226e-18  5.88586113421955e-18  2.28683336101730e-18
0.6   -1.72745751406263e-02  -2.82508935294770e-02  -1.09763183888507e-02
0.7   -2.79508497187474e-02  -4.57109059432483e-02  -1.77600562245009e-02

```

0.8	-2.79508497187474e-02	-4.57109059432483e-02	-1.77600562245009e-02
0.9	-1.72745751406263e-02	-2.82508935294770e-02	-1.09763183888507e-02
1	0.00000000000000e+00	-1.17717222684391e-17	-1.17717222684391e-17

```
# K=10 N=2 T=0.1 NU=0.1
# dx=0.1 dt=0.05 rho=0.5
# SCHEME=Midpoint
```

#	xk	uk	vk	vk-uk
	0	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00
	0.1	3.11237285610348e-02	2.82508935294770e-02	-2.87283503155776e-03
	0.2	5.03592506683801e-02	4.57109059432483e-02	-4.64834472513182e-03
	0.3	5.03592506683801e-02	4.57109059432483e-02	-4.64834472513182e-03
	0.4	3.11237285610348e-02	2.82508935294770e-02	-2.87283503155776e-03
	0.5	4.08612428990798e-18	5.88586113421955e-18	1.79973684431158e-18
	0.6	-3.11237285610348e-02	-2.82508935294770e-02	2.87283503155775e-03
	0.7	-5.03592506683801e-02	-4.57109059432483e-02	4.64834472513183e-03
	0.8	-5.03592506683801e-02	-4.57109059432483e-02	4.64834472513183e-03
	0.9	-3.11237285610348e-02	-2.82508935294770e-02	2.87283503155776e-03
	1	0.00000000000000e+00	-1.17717222684391e-17	-1.17717222684391e-17

```
# K=10 N=18 T=0.9 NU=0.1
# dx=0.1 dt=0.05 rho=0.5
# SCHEME=Left
```

#	xk	uk	vk	vk-uk
	0	0.00000000000000e+00	-0.00000000000000e+00	-0.00000000000000e+00
	0.1	-1.43487837117188e-02	-2.46565476862823e-02	-1.03077639745635e-02
	0.2	-2.32168197427819e-02	-3.98951322016374e-02	-1.66783124588554e-02
	0.3	-2.32168197427819e-02	-3.98951322016374e-02	-1.66783124588554e-02
	0.4	-1.43487837117188e-02	-2.46565476862823e-02	-1.03077639745635e-02
	0.5	-9.29281593288540e-18	-5.13700621820321e-18	4.15580971468219e-18
	0.6	1.43487837117188e-02	2.46565476862823e-02	1.03077639745635e-02
	0.7	2.32168197427819e-02	3.98951322016374e-02	1.66783124588554e-02
	0.8	2.32168197427819e-02	3.98951322016374e-02	1.66783124588554e-02
	0.9	1.43487837117188e-02	2.46565476862823e-02	1.03077639745635e-02
	1	0.00000000000000e+00	1.02740124364064e-17	1.02740124364064e-17

```
# K=10 N=18 T=0.9 NU=0.1
# dx=0.1 dt=0.05 rho=0.5
# SCHEME=Midpoint
```

#	xk	uk	vk	vk-uk
	0	0.00000000000000e+00	-0.00000000000000e+00	-0.00000000000000e+00
	0.1	-2.83411355572518e-02	-2.46565476862823e-02	3.68458787096949e-03
	0.2	-4.58569206114016e-02	-3.98951322016374e-02	5.96178840976424e-03
	0.3	-4.58569206114016e-02	-3.98951322016374e-02	5.96178840976425e-03
	0.4	-2.83411355572518e-02	-2.46565476862823e-02	3.68458787096949e-03
	0.5	-1.30619518481369e-17	-5.13700621820321e-18	7.92494562993373e-18
	0.6	2.83411355572518e-02	2.46565476862823e-02	-3.68458787096949e-03
	0.7	4.58569206114016e-02	3.98951322016374e-02	-5.96178840976424e-03
	0.8	4.58569206114016e-02	3.98951322016374e-02	-5.96178840976425e-03
	0.9	2.83411355572518e-02	2.46565476862823e-02	-3.68458787096949e-03
	1	0.00000000000000e+00	1.02740124364064e-17	1.02740124364064e-17


```

# K=10 N=40 T=2 NU=0.1
# dx=0.1 dt=0.05 rho=0.5
# SCHEME=Left
#  xk                uk                vk                vk-uk
  0    0.00000000000000e+00    -0.00000000000000e+00    -0.00000000000000e+00
0.1   -4.99895941924932e-02    -4.25568546447194e-02     7.43273954777381e-03
0.2   -8.08848624872684e-02    -6.88584372694449e-02     1.20264252178236e-02
0.3   -8.08848624872684e-02    -6.88584372694449e-02     1.20264252178236e-02
0.4   -4.99895941924933e-02    -4.25568546447194e-02     7.43273954777382e-03
0.5   -7.06764291188987e-18    -8.86640050823986e-18    -1.79875759634999e-18
0.6    4.99895941924933e-02     4.25568546447194e-02    -7.43273954777381e-03
0.7    8.08848624872684e-02     6.88584372694449e-02    -1.20264252178236e-02
0.8    8.08848624872684e-02     6.88584372694449e-02    -1.20264252178236e-02
0.9    4.99895941924933e-02     4.25568546447194e-02    -7.43273954777381e-03
  1    0.00000000000000e+00     1.77328010164797e-17     1.77328010164797e-17

```

```

# K=10 N=40 T=2 NU=0.1
# dx=0.1 dt=0.05 rho=0.5
# SCHEME=Midpoint
#  xk                uk                vk                vk-uk
  0    0.00000000000000e+00    -0.00000000000000e+00    -0.00000000000000e+00
0.1   -4.75429293037211e-02    -4.25568546447194e-02     4.98607465900163e-03
0.2   -7.69260755381541e-02    -6.88584372694449e-02     8.06763826870918e-03
0.3   -7.69260755381541e-02    -6.88584372694449e-02     8.06763826870918e-03
0.4   -4.75429293037211e-02    -4.25568546447194e-02     4.98607465900163e-03
0.5   -1.89213925526808e-18    -8.86640050823986e-18    -6.97426125297178e-18
0.6    4.75429293037211e-02     4.25568546447194e-02    -4.98607465900162e-03
0.7    7.69260755381541e-02     6.88584372694449e-02    -8.06763826870918e-03
0.8    7.69260755381541e-02     6.88584372694449e-02    -8.06763826870920e-03
0.9    4.75429293037211e-02     4.25568546447194e-02    -4.98607465900162e-03
  1    0.00000000000000e+00     1.77328010164797e-17     1.77328010164797e-17

```

```

# K=20 N=10 T=0.1 NU=0.1
# dx=0.05 dt=0.01 rho=0.4
# SCHEME=Left
#  xk                uk                vk                vk-uk
  0    0.00000000000000e+00    0.00000000000000e+00    0.00000000000000e+00
0.05   1.39026556150158e-02     1.48523736736112e-02     9.49718058595349e-04
0.1    2.64444224329364e-02     2.82508935294770e-02     1.80647109654057e-03
0.15   3.63976249339960e-02     3.88840190911282e-02     2.48639415713219e-03
0.2    4.27879743093513e-02     4.57109059432483e-02     2.92293163389695e-03
0.25   4.49899386379604e-02     4.80632908350341e-02     3.07335219707367e-03
0.3    4.27879743093513e-02     4.57109059432483e-02     2.92293163389695e-03
0.35   3.63976249339960e-02     3.88840190911282e-02     2.48639415713219e-03
0.4    2.64444224329364e-02     2.82508935294770e-02     1.80647109654058e-03
0.45   1.39026556150158e-02     1.48523736736112e-02     9.49718058595351e-04
0.5    -3.52638561434571e-19     5.88586113421955e-18     6.23849969565412e-18
0.55   -1.39026556150158e-02    -1.48523736736112e-02    -9.49718058595356e-04
0.6    -2.64444224329364e-02    -2.82508935294770e-02    -1.80647109654058e-03
0.65   -3.63976249339960e-02    -3.88840190911282e-02    -2.48639415713218e-03
0.7    -4.27879743093513e-02    -4.57109059432483e-02    -2.92293163389695e-03
0.75   -4.49899386379604e-02    -4.80632908350341e-02    -3.07335219707368e-03

```

0.8	-4.27879743093513e-02	-4.57109059432483e-02	-2.92293163389695e-03
0.85	-3.63976249339960e-02	-3.88840190911282e-02	-2.48639415713218e-03
0.9	-2.64444224329364e-02	-2.82508935294770e-02	-1.80647109654059e-03
0.95	-1.39026556150158e-02	-1.48523736736112e-02	-9.49718058595356e-04
1	0.00000000000000e+00	-1.17717222684391e-17	-1.17717222684391e-17

```
# K=20 N=10 T=0.1 NU=0.1
# dx=0.05 dt=0.01 rho=0.4
# SCHEME=Midpoint
```

#	xk	uk	vk	vk-uk
0	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00	0.00000000000000e+00
0.05	1.51297961830220e-02	1.48523736736112e-02	-2.77422509410789e-04	-2.77422509410789e-04
0.1	2.87785825001612e-02	2.82508935294770e-02	-5.27688970684174e-04	-5.27688970684174e-04
0.15	3.96103206500099e-02	3.88840190911282e-02	-7.26301558881731e-04	-7.26301558881731e-04
0.2	4.65647246333037e-02	4.57109059432483e-02	-8.53818690055436e-04	-8.53818690055436e-04
0.25	4.89610489339759e-02	4.80632908350341e-02	-8.97758098941888e-04	-8.97758098941888e-04
0.3	4.65647246333037e-02	4.57109059432483e-02	-8.53818690055436e-04	-8.53818690055436e-04
0.35	3.96103206500099e-02	3.88840190911282e-02	-7.26301558881731e-04	-7.26301558881731e-04
0.4	2.87785825001612e-02	2.82508935294770e-02	-5.27688970684170e-04	-5.27688970684170e-04
0.45	1.51297961830220e-02	1.48523736736112e-02	-2.77422509410787e-04	-2.77422509410787e-04
0.5	-1.58935183328642e-19	5.88586113421955e-18	6.04479631754820e-18	6.04479631754820e-18
0.55	-1.51297961830220e-02	-1.48523736736112e-02	2.77422509410780e-04	2.77422509410780e-04
0.6	-2.87785825001612e-02	-2.82508935294770e-02	5.27688970684163e-04	5.27688970684163e-04
0.65	-3.96103206500099e-02	-3.88840190911282e-02	7.26301558881738e-04	7.26301558881738e-04
0.7	-4.65647246333037e-02	-4.57109059432483e-02	8.53818690055436e-04	8.53818690055436e-04
0.75	-4.89610489339759e-02	-4.80632908350341e-02	8.97758098941888e-04	8.97758098941888e-04
0.8	-4.65647246333037e-02	-4.57109059432483e-02	8.53818690055429e-04	8.53818690055429e-04
0.85	-3.96103206500099e-02	-3.88840190911282e-02	7.26301558881745e-04	7.26301558881745e-04
0.9	-2.87785825001612e-02	-2.82508935294770e-02	5.27688970684163e-04	5.27688970684163e-04
0.95	-1.51297961830220e-02	-1.48523736736112e-02	2.77422509410782e-04	2.77422509410782e-04
1	0.00000000000000e+00	-1.17717222684391e-17	-1.17717222684391e-17	-1.17717222684391e-17

```
# K=20 N=90 T=0.9 NU=0.1
# dx=0.05 dt=0.01 rho=0.4
# SCHEME=Left
```

#	xk	uk	vk	vk-uk
0	0.00000000000000e+00	-0.00000000000000e+00	-0.00000000000000e+00	-0.00000000000000e+00
0.05	-1.20424883029356e-02	-1.29627142361276e-02	-9.20225933192017e-04	-9.20225933192017e-04
0.1	-2.29061739458302e-02	-2.46565476862823e-02	-1.75037374045211e-03	-1.75037374045211e-03
0.15	-3.15276436862085e-02	-3.39368264566343e-02	-2.40918277042579e-03	-2.40918277042579e-03
0.2	-3.70629679965706e-02	-3.98951322016374e-02	-2.83216420506681e-03	-2.83216420506681e-03
0.25	-3.89703107665458e-02	-4.19482244410133e-02	-2.97791367446755e-03	-2.97791367446755e-03
0.3	-3.70629679965706e-02	-3.98951322016374e-02	-2.83216420506681e-03	-2.83216420506681e-03
0.35	-3.15276436862085e-02	-3.39368264566343e-02	-2.40918277042579e-03	-2.40918277042579e-03
0.4	-2.29061739458302e-02	-2.46565476862823e-02	-1.75037374045212e-03	-1.75037374045212e-03
0.45	-1.20424883029356e-02	-1.29627142361277e-02	-9.20225933192024e-04	-9.20225933192024e-04
0.5	7.93880769078472e-18	-5.13700621820321e-18	-1.30758139089879e-17	-1.30758139089879e-17
0.55	1.20424883029356e-02	1.29627142361277e-02	9.20225933192013e-04	9.20225933192013e-04
0.6	2.29061739458302e-02	2.46565476862823e-02	1.75037374045211e-03	1.75037374045211e-03
0.65	3.15276436862085e-02	3.39368264566343e-02	2.40918277042578e-03	2.40918277042578e-03
0.7	3.70629679965706e-02	3.98951322016374e-02	2.83216420506680e-03	2.83216420506680e-03
0.75	3.89703107665458e-02	4.19482244410133e-02	2.97791367446754e-03	2.97791367446754e-03
0.8	3.70629679965706e-02	3.98951322016374e-02	2.83216420506680e-03	2.83216420506680e-03

0.85	3.15276436862085e-02	3.39368264566343e-02	2.40918277042577e-03
0.9	2.29061739458302e-02	2.46565476862823e-02	1.75037374045212e-03
0.95	1.20424883029356e-02	1.29627142361277e-02	9.20225933192020e-04
1	0.00000000000000e+00	1.02740124364064e-17	1.02740124364064e-17

K=20 N=90 T=0.9 NU=0.1
dx=0.05 dt=0.01 rho=0.4
SCHEME=Midpoint

#	xk	uk	vk	vk-uk
0	0.00000000000000e+00	-0.00000000000000e+00	-0.00000000000000e+00	-0.00000000000000e+00
0.05	-1.33024402287967e-02	-1.29627142361276e-02	3.39725992669061e-04	3.39725992669061e-04
0.1	-2.53027449244478e-02	-2.46565476862823e-02	6.46197238165502e-04	6.46197238165502e-04
0.15	-3.48262406523037e-02	-3.39368264566343e-02	8.89414195669402e-04	8.89414195669402e-04
0.2	-4.09407012964254e-02	-3.98951322016374e-02	1.04556909478809e-03	1.04556909478809e-03
0.25	-4.30476008470140e-02	-4.19482244410133e-02	1.09937640600068e-03	1.09937640600068e-03
0.3	-4.09407012964254e-02	-3.98951322016374e-02	1.04556909478809e-03	1.04556909478809e-03
0.35	-3.48262406523037e-02	-3.39368264566343e-02	8.89414195669402e-04	8.89414195669402e-04
0.4	-2.53027449244478e-02	-2.46565476862823e-02	6.46197238165506e-04	6.46197238165506e-04
0.45	-1.33024402287967e-02	-1.29627142361277e-02	3.39725992669064e-04	3.39725992669064e-04
0.5	-6.56083137450032e-18	-5.13700621820321e-18	1.42382515629712e-18	1.42382515629712e-18
0.55	1.33024402287967e-02	1.29627142361277e-02	-3.39725992669047e-04	-3.39725992669047e-04
0.6	2.53027449244478e-02	2.46565476862823e-02	-6.46197238165488e-04	-6.46197238165488e-04
0.65	3.48262406523037e-02	3.39368264566343e-02	-8.89414195669389e-04	-8.89414195669389e-04
0.7	4.09407012964254e-02	3.98951322016374e-02	-1.04556909478808e-03	-1.04556909478808e-03
0.75	4.30476008470140e-02	4.19482244410133e-02	-1.09937640600066e-03	-1.09937640600066e-03
0.8	4.09407012964254e-02	3.98951322016374e-02	-1.04556909478808e-03	-1.04556909478808e-03
0.85	3.48262406523037e-02	3.39368264566343e-02	-8.89414195669402e-04	-8.89414195669402e-04
0.9	2.53027449244478e-02	2.46565476862823e-02	-6.46197238165488e-04	-6.46197238165488e-04
0.95	1.33024402287967e-02	1.29627142361277e-02	-3.39725992669054e-04	-3.39725992669054e-04
1	0.00000000000000e+00	1.02740124364064e-17	1.02740124364064e-17	1.02740124364064e-17

K=20 N=200 T=2 NU=0.1
dx=0.05 dt=0.01 rho=0.4
SCHEME=Left

#	xk	uk	vk	vk-uk
0	0.00000000000000e+00	-0.00000000000000e+00	-0.00000000000000e+00	-0.00000000000000e+00
0.05	-2.32029785933465e-02	-2.23734625206607e-02	8.29516072685874e-04	8.29516072685874e-04
0.1	-4.41346879773184e-02	-4.25568546447194e-02	1.57783333259893e-03	1.57783333259893e-03
0.15	-6.07461865976175e-02	-5.85744853251115e-02	2.17170127250593e-03	2.17170127250593e-03
0.2	-7.14114252301725e-02	-6.88584372694449e-02	2.55298796072757e-03	2.55298796072757e-03
0.25	-7.50864160085418e-02	-7.24020456089017e-02	2.68437039964013e-03	2.68437039964013e-03
0.3	-7.14114252301725e-02	-6.88584372694449e-02	2.55298796072757e-03	2.55298796072757e-03
0.35	-6.07461865976175e-02	-5.85744853251115e-02	2.17170127250593e-03	2.17170127250593e-03
0.4	-4.41346879773184e-02	-4.25568546447194e-02	1.57783333259892e-03	1.57783333259892e-03
0.45	-2.32029785933465e-02	-2.23734625206607e-02	8.29516072685863e-04	8.29516072685863e-04
0.5	8.72181355458280e-18	-8.86640050823986e-18	-1.75882140628227e-17	-1.75882140628227e-17
0.55	2.32029785933466e-02	2.23734625206607e-02	-8.29516072685870e-04	-8.29516072685870e-04
0.6	4.41346879773184e-02	4.25568546447194e-02	-1.57783333259894e-03	-1.57783333259894e-03
0.65	6.07461865976175e-02	5.85744853251115e-02	-2.17170127250597e-03	-2.17170127250597e-03
0.7	7.14114252301725e-02	6.88584372694449e-02	-2.55298796072760e-03	-2.55298796072760e-03
0.75	7.50864160085419e-02	7.24020456089017e-02	-2.68437039964016e-03	-2.68437039964016e-03
0.8	7.14114252301725e-02	6.88584372694449e-02	-2.55298796072760e-03	-2.55298796072760e-03
0.85	6.07461865976175e-02	5.85744853251115e-02	-2.17170127250599e-03	-2.17170127250599e-03

0.9	4.41346879773184e-02	4.25568546447194e-02	-1.57783333259892e-03
0.95	2.32029785933465e-02	2.23734625206607e-02	-8.29516072685860e-04
1	0.00000000000000e+00	1.77328010164797e-17	1.77328010164797e-17

```
# K=20 N=200 T=2 NU=0.1
# dx=0.05 dt=0.01 rho=0.4
# SCHEME=Midpoint
```

#	xk	uk	vk	vk-uk
0	0.00000000000000e+00	-0.00000000000000e+00	-0.00000000000000e+00	-0.00000000000000e+00
0.05	-2.27937029174531e-02	-2.23734625206607e-02	4.20240396792389e-04	4.20240396792389e-04
0.1	-4.33561993802792e-02	-4.25568546447194e-02	7.99344735559718e-04	7.99344735559718e-04
0.15	-5.96746889673597e-02	-5.85744853251115e-02	1.10020364224821e-03	1.10020364224821e-03
0.2	-7.01518042203088e-02	-6.88584372694449e-02	1.29336695086392e-03	1.29336695086392e-03
0.25	-7.37619720998134e-02	-7.24020456089017e-02	1.35992649091166e-03	1.35992649091166e-03
0.3	-7.01518042203088e-02	-6.88584372694449e-02	1.29336695086392e-03	1.29336695086392e-03
0.35	-5.96746889673597e-02	-5.85744853251115e-02	1.10020364224821e-03	1.10020364224821e-03
0.4	-4.33561993802792e-02	-4.25568546447194e-02	7.99344735559711e-04	7.99344735559711e-04
0.45	-2.27937029174531e-02	-2.23734625206607e-02	4.20240396792379e-04	4.20240396792379e-04
0.5	4.64438223495781e-18	-8.86640050823986e-18	-1.35107827431977e-17	-1.35107827431977e-17
0.55	2.27937029174531e-02	2.23734625206607e-02	-4.20240396792376e-04	-4.20240396792376e-04
0.6	4.33561993802792e-02	4.25568546447194e-02	-7.99344735559718e-04	-7.99344735559718e-04
0.65	5.96746889673597e-02	5.85744853251115e-02	-1.10020364224823e-03	-1.10020364224823e-03
0.7	7.01518042203088e-02	6.88584372694449e-02	-1.29336695086393e-03	-1.29336695086393e-03
0.75	7.37619720998134e-02	7.24020456089017e-02	-1.35992649091166e-03	-1.35992649091166e-03
0.8	7.01518042203088e-02	6.88584372694449e-02	-1.29336695086392e-03	-1.29336695086392e-03
0.85	5.96746889673597e-02	5.85744853251115e-02	-1.10020364224824e-03	-1.10020364224824e-03
0.9	4.33561993802792e-02	4.25568546447194e-02	-7.99344735559704e-04	-7.99344735559704e-04
0.95	2.27937029174531e-02	2.23734625206607e-02	-4.20240396792372e-04	-4.20240396792372e-04
1	0.00000000000000e+00	1.77328010164797e-17	1.77328010164797e-17	1.77328010164797e-17

Note that our numeric output verifies that the midpoint is generally more accurate than the left endpoint for approximating the contribution of the force $F(x, t)$.