#1 Prove that $((B_n(f)(x))^2 \leq (B_n(f^2))(x)$ for $x \in [0,1]$.

Recall the Cauchy-Schwartz inequality

$$\left|\sum_{k=0}^{n} a_k b_k\right| \leq \sqrt{\sum_{k=0}^{n} a_k^2}\sqrt{\sum_{k=0}^{n} b_k^2}.$$

Recall also that

$$\sum_{k=0}^{n} P_k^n(x) = 1 \quad \text{where} \quad P_k^n(x) = \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k}.$$

Since $x \in [0,1]$ then $P_k^n(x) \geq 0$.

By the Cauchy-Schwartz inequality

$$\begin{aligned}
|(B_n(f))(x)| &= \left|\sum_{k=0}^{n} f\left(\tfrac{k}{n}\right) P_k^n(x)\right| \\
&= \left|\sum_{k=0}^{n} f\left(\tfrac{k}{n}\right)\sqrt{P_k^n(x)} \cdot \sqrt{P_k^n(x)}\right| \\
&\leq \sqrt{\sum_{k=0}^{n}\left(f\left(\tfrac{k}{n}\right)\right)^2 P_k^n(x)}\ \sqrt{\sum_{k=0}^{n} P_k^n(x)} \\
&= \sqrt{(B_n(f^2))(x)}
\end{aligned}$$

Therefore

$$((B_n(f))(x))^2 \leq (B_n(f^2))(x).$$

#2 Let $U, V \subseteq \mathbb{R}$ be open subsets. Prove or disprove $\overline{U} \cap \overline{V} = \overline{U \cap V}$.

Counterexample

$U = (0,1) \qquad V = (1,2)$

then $\overline{U} = [0,1]$. $\overline{V} = [1,2]$ so $\overline{U} \cap \overline{V} = \{1\}$.

However $U \cap V = \emptyset$ so $\overline{U \cap V} = \emptyset$.

#3 Let $\mathcal{E}, \mathcal{F}$ be collections of subsets of $\mathbb{R}$. Let $\mathcal{A}(\mathcal{E})$ be the σ-algebra generated by $\mathcal{E}$, $\mathcal{A}(\mathcal{F})$ the σ-algebra generated by $\mathcal{F}$ and $\mathcal{A}(\mathcal{E} \cap \mathcal{F})$ be the σ-algebra generated by $\mathcal{E} \cap \mathcal{F}$. Prove or disprove $\mathcal{A}(\mathcal{E}) \cap \mathcal{A}(\mathcal{F}) = \mathcal{A}(\mathcal{E} \cap \mathcal{F})$.

<u>Counter example</u>

$$\mathcal{E} = \{ [0,1] \}$$

$$\mathcal{F} = \{ [0,1]^c \}.$$

Then $\mathcal{A}(\mathcal{E}) = \{ \emptyset, [0,1], [0,1]^c, \mathbb{R} \}$

$\mathcal{A}(\mathcal{F}) = \{ \emptyset, [0,1], [0,1]^c, \mathbb{R} \}$

so $\mathcal{A}(\mathcal{E}) \cap \mathcal{A}(\mathcal{F}) = \{ \emptyset, [0,1], [0,1]^c, \mathbb{R} \}$.

However

$$\mathcal{A}(\mathcal{E} \cap \mathcal{F}) = \mathcal{A}(\emptyset) = \{ \emptyset, \mathbb{R} \}.$$

#5. For subsets $A, B \subseteq \mathbb{R}$ define $A+B = \{a+b : a \in A \text{ and } b \in B\}$. Suppose B is a Borel set.

(a) Prove $A+B$ is a Borel set if A is countable.

<u>Lemma</u>: If $f, g \in \hat{C}$ then $f \circ g \in \hat{C}$.

<u>proof</u>: We will use the characterization that

$$\hat{C} = \mathcal{F}_1 = \{f : \mathbb{R} \to \mathbb{R} \text{ st. } f^{-1}(B) \in \mathcal{B} \text{ for all } B \in \mathcal{B}\}.$$

which makes this lemma obvious.

Let $B \in \mathcal{B}$. Then $f^{-1}(B) \in \mathcal{B}$ since $f \in \hat{C}$. Thus

$$(f \circ g)^{-1}(B) = g^{-1}(f^{-1}(B)) \in \mathcal{B} \text{ since } g \in \hat{C}.$$

This implies $f \circ g \in \hat{C}$.

<u>Lemma</u>: If $B \in \mathcal{B}$ then $x + B \in \mathcal{B}$ for every $x \in \mathbb{R}$.

<u>Proof</u>: Let $x \in \mathbb{R}$ and define $g(z) = z - x$. Then g is continuous and therefore $g \in \hat{C}$. Define $f = \chi_B$. Since $B \in \mathcal{B}$ then $f \in \hat{C}$. It follows that $f \circ g \in \hat{C}$. Thus

$$(f \circ g)(z) = \chi_B(g(z)) = \chi_B(z - x) = \chi_{x+B}(z)$$

shows that $\chi_{x+B} \in \hat{C}$.

Therefore $x + B \in \mathcal{B}$.

We now use these lemmas to prove (a)

<u>Proof of (a)</u>

Since A is countable we may enumerate the elements of A as $\{a_1, a_2, a_3, \ldots\}$.

Thus

$$A+B = \bigcup (a_n + B)$$

Since $a_n + B \in \mathcal{B}$ then $A+B$ is a countable union of sets in $\mathcal{B}$. Since $\mathcal{B}$ is a σ-algebra it follows that $A+B \in \mathcal{B}$.

(b) Prove that if A is open and B is Borel that $A+B$ is Borel.

Lemma If $A \in \tau$ then $A+x \in \tau$ for every $x \in \mathbb{R}$.

Proof: Let $z \in A+x$. Therefore $z-x \in A$. Since A is open there is $r>0$ such that

$|z-x-y|<r$ implies $y \in A$.

Now

$|z-w|<r$ implies

$|z-x-(w-x)|<r$ implies

$|z-x-y|<r$ where $y=w-x$.

Therefore

$y \in A$ implies

$w-x \in A$ implies

$w \in A+x$

It follows that $A+x$ is open.

We now use this lemma to prove (b).

Proof of (b) Since

$$A+B = \bigcup_{x\in B}(A+x)$$

and $A+x$ is open, then $A+B$ is a union of open sets. Since any union of open sets is open then $A+B$ is open.

Therefore

$$A+B \in \tau \subseteq \mathcal{B}.$$

#6. Let I be any finite interval. Show for each $\varepsilon>0$ there is a closed interval $J\subseteq I$ such that $\ell(I)<\ell(J)+\varepsilon$.

Let $\varepsilon>0$. If $\ell(I)<\varepsilon$. Then take $J=[x,x]$ where x is some point in I. Then

$$\ell(I)<\varepsilon=0+\varepsilon=\ell(J)+\varepsilon.$$

Otherwise I must be of the form

$$(a,b)\,,\ (a,b]\,,\ [a,b)\ \text{or}\ [a,b]$$

where $a,b\in\mathbb{R}$ and $a<b$.

In each of these cases $(a,b)\subseteq I$.

Claim: There is a closed interval $J\subset(a,b)$ with $b-a<\ell(J)+\varepsilon$.

Since $b-a=\ell(I)\geqslant\varepsilon$ then $a+\frac{\varepsilon}{3}<b-\frac{\varepsilon}{3}$.

Therefore $J=[a+\frac{\varepsilon}{3},b-\frac{\varepsilon}{3}]$ is a closed interval.

Moreover

$$J=\left[a+\tfrac{\varepsilon}{3},b-\tfrac{\varepsilon}{3}\right]\subseteq(a,b)\subseteq I$$

and

$$\ell(J)=a+\tfrac{\varepsilon}{3}-\left(b-\tfrac{\varepsilon}{3}\right)=a-b+\tfrac{2}{3}\varepsilon$$

$$<a-b+\varepsilon=\ell(I)+\varepsilon.$$