#2.52 In this exercise, we will explore limit points, closure and closed sets of a subset of $\mathbb{R}$. Let $D \subseteq \mathbb{R}$ and $E \subseteq D$.

(a) Define a limit point of E relative to D. Call such a limit point a D-limit point of E.

$x \in D$ is a D-limit point of E if for every $\varepsilon > 0$ there is a $y \in E$ such that $|y - x| < \varepsilon$.

(b) Define the closure of E in D. Call it the D-closure of E.

The D-closure of E is the set of all D-limit points of E.

$$\overline{E}^D = \{x \in D : x \text{ is a } D\text{-limit point}\}.$$

(c) Define E closed in D.

E is closed in D if $\overline{E}^D = E$.

(d) Prove E is closed in D if and only if $D \setminus E$ is open in D.

"$\Longrightarrow$" Suppose E is closed in D. Let $z \in D \setminus E$.
Claim there is $r > 0$ such that $(z-r, z+r) \cap D \subseteq D \setminus E$.
If not then there is a sequence $r_n = \frac{1}{n}$ such that $(z - r_n, z + r_n) \cap D$ is not contained in $D \setminus E$ for every $n \in \mathbb{N}$.
Thus there is $x_n \in E \cap (z - r_n, z + r_n)$ for every $n \in \mathbb{N}$.
Since $x_n \to z$ we have that $z \in \overline{E}$ and since E is closed then $z \in E$. This contradicts $z \in D \setminus E$.
Therefore $D \setminus E$ is open in D.

"$\Leftarrow$" Suppose $D\setminus E$ is open in D. Claim that E is closed in D. Suppose not. Then there exists $x \in D\setminus E$ such that x is a D-limit point of E. Since $D\setminus E$ is open there is $r>0$ such that $(x-r, x+r)\cap D \subseteq D\setminus E$. Choose $\varepsilon = r$. Then since x is a D-limit point of E there is $y \in E$ such that $|x-y|<\varepsilon = r$. Therefore

$$\begin{aligned} y \in (x-r, x+r) &\subseteq (x-r, x+r)\cap E \\ &\subseteq (x-r, x+r)\cap D \cap E \\ &\subseteq (D\setminus E)\cap E = \emptyset \end{aligned}$$

which is a contradiction. Therefore E is closed in D.

(e) Prove that E is closed in D if and only there is a closed set F of $\mathbb{R}$ such that $E = D\cap F$.

"$\Rightarrow$" Suppose E is closed in D. Let $F = \overline{E}$. Then F is closed. Moreover

$$\begin{aligned} E = \overline{E}^{D} &= \{x \in D : \forall \varepsilon>0 \ \exists y \in E \text{ s.t. } |y-x|<\varepsilon\} \\ &= D\cap \{x\in\mathbb{R} : \forall \varepsilon>0 \ \exists y\in E \text{ s.t. } |y-x|<\varepsilon\} \\ &= D\cap \overline{E} = D\cap F. \end{aligned}$$

"$\Leftarrow$" Suppose there is closed set F of $\mathbb{R}$ such that $E = D\cap F$. Let $x\in D$ be a D-limit point of E. Then for every $\varepsilon>0$ there is $y\in E$ such that $|y-x|<\varepsilon$. Since $E\subseteq F$ then x is also a limit point of F. Since F is closed then $x\in F$. Therefore $x\in D\cap F = E$, which implies that E is closed relative to D.