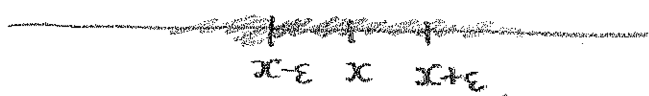
Math 713 Summary Sept 15, '10

Structure theorem of the Real Numbers

Let $U \subseteq \mathbb{R}$ be open. Then U may be written as a countable union of disjoint intervals.

<u>Proof</u>: Let $x \in U$. Since U is open then there is some $\varepsilon > 0$ so that $(x-\varepsilon, x+\varepsilon) \subseteq U$.



The idea is to increase the upper and lower endpoints to be as large and as small as possible.

Define

$$a_x = \inf \{ a : (a,x) \subseteq U \} = \inf A_x .$$

$$b_x = \sup \{ b : (x,b) \subseteq U \} = \sup B_x .$$

and define

$$I_x = (a_x, b_x) \quad \text{for } x \in U .$$

Note that A_x is non empty since $x-\varepsilon \in A_x$, and B_x is non empty since $x+\varepsilon \in B_x$. There fore I_x is well defined,

The proof procedes with the following claims:

1. $\bigcup_{x \in U} I_x = U$

   (a) $x \in I_x$

   (b) $I_x \subseteq U$

2. $\{I_x : x \in U\}$ is disjoint

   (a) $a_x \notin U$ and $b_x \notin U$

   (b) If $I_x \cap I_y \neq \emptyset$ then $a_x = a_y$ and $b_x = b_y$.

3. $\{I_x : x \in U\}$ is countable.

4. The set $\{I_x : x \in U\}$ is unique.

Before continuing with the proof lets have a vote...

Let $A \subseteq \mathbb{R}$. True or false that $\inf A \leq \sup A$?

It seem reasonable since $\inf A = \min A$ and $\sup A = \max A$ in the case where the minimum and maximum exist.

Suppose $A \subseteq \mathbb{R}$. True of false if A is closed and bounded then minimum of A exists and $\inf A = \min A$?

A possible example $A = [0,1]$ then $\min A = 0$. So is this true for all closed and bounded sets?

True or false, the empty set $\emptyset$ is closed and bounded.

$M=1$ is a bound since if $x \in \emptyset$ then $|x| \leq 1$. By Theorem 2.2 $\mathbb{R}$ and $\emptyset$ are open. Therefore $\emptyset = \mathbb{R}^c$ and $\mathbb{R} = \emptyset^c$ are also closed.

Obviously $\emptyset$ has no minimal element because it has no elements at all. Thus $\min \emptyset$ is not defined.

What about $\inf \emptyset$?

Is 1 a lower bound?

Yes, for every $x \in \emptyset$ we have $x \geq 1$

Is 2 a lower bound?

Yes, for every $x \in \emptyset$ we have $x \geq 2$.

Thus $\inf \emptyset = \infty$.

Similarly $\sup \emptyset = -\infty$.

Take $A = \emptyset \subseteq \mathbb{R}$ for an example of a set such that

$$\inf A > \sup A.$$

Now, back to the proof of the structure theorem.

Since $A_x \neq \emptyset$ and $B_x \neq \emptyset$ for $x \in U$, then none of this strange stuff with infimum and supremum of $\emptyset$ happens in the definition of a_x, b_x and I_x.

<u>Claim 1a.</u> $x \in I_x$.

Since $x-\varepsilon \in A_x$ then $a_x \leq x-\varepsilon$ and so $a_x < x$.
Since $x+\varepsilon \in B_x$ then $b_x \geq x+\varepsilon$ and so $b_x > x$.
Therefore $a_x < x < b$ or in other words $x \in I_x$

<u>Claim 1b.</u> $I_x \subseteq U$.

Let $y \in I_x$. Need to show $y \in U$.
Case $y = x$. Then $y = x \in U$ by assumption.
Case $y < x$. Then $y \in I_x = (a_x, b_x)$ implies that $a_x < y < x$. Since a_x is the greatest lower bound and y is greater than a_x then y can't be a lower bound for A_x. It follows that there is $a \in A_x$ such that $a < y$. Then $a \in A_x$ implies $(a, x) \subseteq U$ and so $y \in (a, x) \subseteq U$.

Claim 1b continued...

Case $y > x$. Then $y \in I_x = (a_x, b_x)$ implies that $x < y < b_x$. Since b_x is the least upper bound and y is less that b_x then y can't be an upper bound for B_x. It follows that there is $b \in B_x$ such that $b > y$. Then $b \in B_x$ implies $(x,b) \subseteq U$ and so $y \in (x,b) \subseteq U$,

Note, in class I introduced a value z that was also not a lower or upper bound. While correct this additional step was unnecessary.

Through claim 1a and 1b we conclude that

$$U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} I_x \subseteq \bigcup_{x \in U} U = U$$

and so $U = \bigcup_{x \in U} I_x$.

Take a 5 minute break,

<u>Claim 2a</u> $a_x \notin U$ and $b_x \notin U$.

For contradiction suppose $a_x \in U$. Then for some $\varepsilon > 0$ we have $(a_x - \varepsilon, a_x + \varepsilon) \subseteq U$. Now since $a_x + \varepsilon > a_x$ and a_x is the greatest lower bound then $a_x + \varepsilon$ is not a lower bound. It follows there is $a \in A_x$ such that $a < a_x + \varepsilon$. and $(a, x) \subseteq U$. Thus,

$$(a_x - \varepsilon, a_x + \varepsilon) \cup (a, x) \subseteq U$$

or

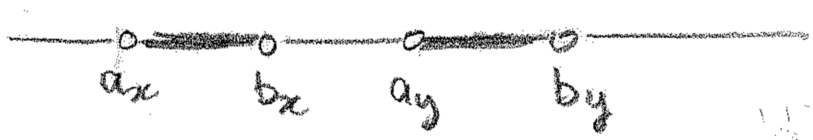
$$(a_x - \varepsilon, x) \subseteq U.$$

Which implies $a_x - \varepsilon \in A_x$. But this contradicts a_x being a lowerbound of A_x.

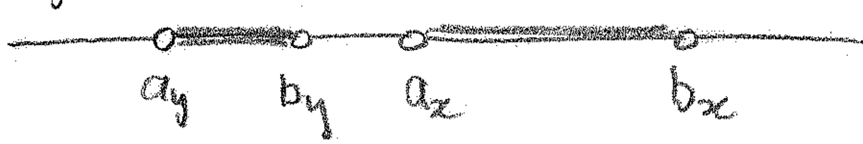
The proof that $b_x \notin U$ is similar.

<u>Claim 2b</u> If $I_x \cap I_y \neq \emptyset$ then $a_x = a_y$ and $b_x = b_y$.

If $I_x \cap I_y = \emptyset$ then either $b_x \leq a_y$ as in



or $b_y \leq a_x$ as in



Thus

$I_x \cap I_y = \emptyset$ means either $b_x \le a_y$ or $b_y \le a_x$.

Equivalently, DeMorgan's law gives

$I_x \cap I_y \ne \emptyset$ means both $a_y < b_x$ and $a_x < b_y$.

Since $a_y \notin U$ then $a_y \notin (a_x, b_x) \subseteq U$.
Therefore a_y is either outside I_x to the right or outside to the left.
Since $a_y < b_x$ then a_y must be outside I_x to the left.
Hence $a_y \le a_x$.

Since $a_x \notin U$ then $a_x \notin (a_y, b_y) \subseteq U$.
Therefore a_x is either outside I_y to the right or outside to the left.
Since $a_x < b_y$ then a_x must be outside I_x to the left.
Hence $a_x \le a_y$.

It follows that $a_x = a_y$.
Similarly $b_x = b_y$.

Claim 3. $\{I_x : x \in U\}$ is countable.

Let $\mathcal{C} = \{I_x : x \in U\}$ and $\mathcal{D} = \{I_x \cap \mathbb{Q} : x \in U\}$.
Define $\tau : \mathcal{C} \to \mathcal{D}$ by $\tau(I_x) = I_x \cap \mathbb{Q}$.
Since $\mathbb{Q}$ is dense then $I_x \cap \mathbb{Q} \neq \emptyset$ for $x \in U$.
Since $\mathcal{C}$ is a collection of disjoint sets then τ is 1-to-1 and therefore a bijection. Thus

$$\mathcal{C} \sim \mathcal{D}.$$

By the axiom of choice there is $w : \mathcal{D} \to \mathbb{Q}$ such that $w(I_x \cap \mathbb{Q}) \in I_x \cap \mathbb{Q}$ forall $I_x \cap \mathbb{Q} \in \mathcal{D}$.
Since $\mathcal{D}$ is a collection of disjoint sets then w is 1-to-1.
Thus

$$\mathcal{D} \sim w(\mathcal{D}) \subseteq \mathbb{Q}$$

In otherwords, $\mathcal{D}$ is set equivalent to a subset of the rational numbers and by proposition 1.8 therefore countable.

It follows that $\mathcal{C}$ is countable.

The final claim of uniqueness will be discussed on Monday