Review Summary

Sept 17, '10

Discussion of homework problem 3.

Let A be an uncountable set and B be a countable subset of A. Show that A is set equivalent to $A \setminus B$.

Example 1.

$A = [0,1]$ $\quad B = \{1\}$.

Then $A \setminus B = [0,1)$.

Let $f : [0,1] \to [0,1)$

$f(x) = x/2$

$g : [0,1) \to [0,1]$

$g(x) = x$

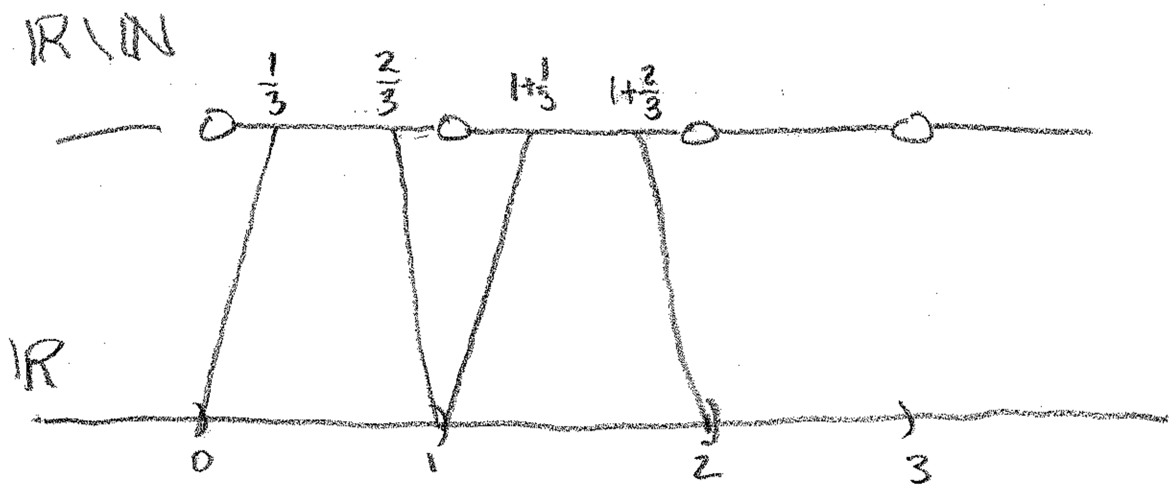
Clearly f and g are one-to-one.

By the Schroeder-Bernstein theorem we have that $[0,1] \sim [0,1)$.

Idea if can't solve the original problem start working on simpler examples. Especially important for research, because often the examples are interesting enough to publish on their own.

Example 2

$$A = \mathbb{R} \qquad B = \mathbb{N}$$



$$f(x) = \begin{cases} \dfrac{1+x}{3} & \text{if } x \in [0,1) \\ \left(1+\frac{1}{3}\right)\left(\dfrac{x-2}{1-2}\right) + \left(1+\frac{2}{3}\right)\left(\dfrac{x-1}{2-1}\right) & \text{if } x \in [1,2) \\ \vdots & \end{cases}$$

General formula:

$$f(x) = \left(n+\frac{1}{3}\right)\left(\frac{x-(n+1)}{n-(n+1)}\right) + \left(n+\frac{2}{3}\right)\left(\frac{x-n}{(n+1)-n}\right) \text{ for } x \in [n, n+1)$$

Then

$$f : \mathbb{R} \to \mathbb{R} \setminus \mathbb{N}$$

$$f([n, n+1)) = \left[n+\frac{1}{3}, n+\frac{2}{3}\right)$$

Example 2 continues...

One should show f is one-to-one, which is more or less obvious because

$$f([n,n+1)) \cap f([m,m+1)) = \emptyset \quad \text{for } n \neq m$$

and on each interval $[n,n+1)$ it is one-to-one.

Taking $g: \mathbb{R} \setminus \mathbb{N} \to \mathbb{R}$ by $g(x)=x$ as before and applying the Schroeder-Bernstein theorem yields that $\mathbb{R} \sim \mathbb{R} \setminus \mathbb{N}$.

Example 3

$$A = \mathbb{R} \qquad B = \mathbb{Q}$$

Ideas?

Hints 1.

$$\mathbb{N} \sim \{2,3,4,5,\ldots\}$$

take $h(n) = n+1$.

Hint 2

$$\mathbb{N} \sim \{2,4,6,8,\ldots\}$$

take $h(n) = 2n$.

Example 3 continued

It's a difficult example, but may build enough intuition to solve the general case.

Idea 1 Since $\mathbb{Q}$ is countable write

$$\mathbb{Q} = \{r_1, r_2, r_3, \dots\} = \{r_n : n \in \mathbb{N}\}.$$

and define

$$A = \{\sqrt{2}r_1, \sqrt{2}r_2, \sqrt{2}r_3, \dots\} = \{\sqrt{2}r_n : n \in \mathbb{N}\}.$$

Then $\mathbb{Q} \cap A = \emptyset$ and we try

$$f(x) = \begin{cases} \sqrt{2}r_n & \text{if } x = r_n \\ x & \text{otherwise.} \end{cases}$$

Is f one-to-one? No.

$$f(\sqrt{2}r_n) = \sqrt{2}r_n = f(r_n)$$

To fix this need to figure out where to map $\sqrt{2}r_n$.

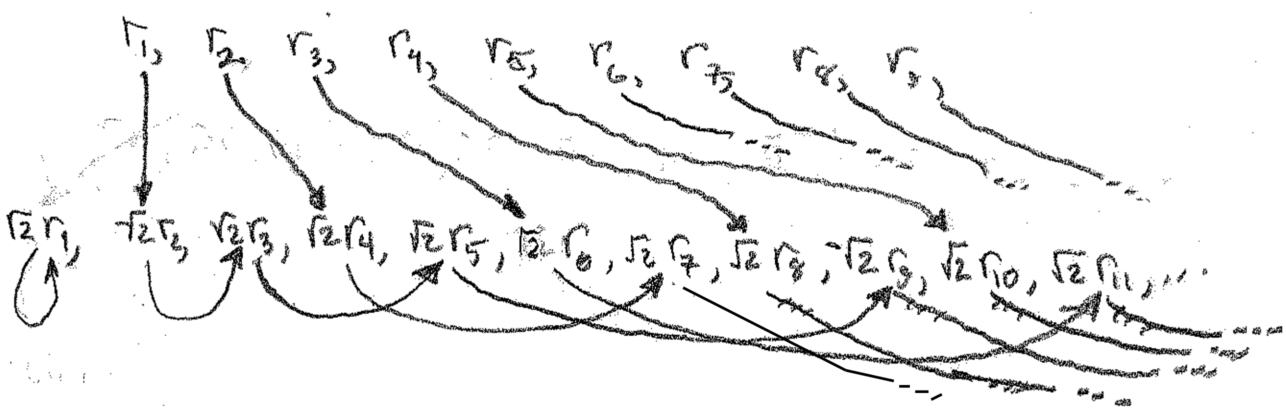
Idea 2:

$$f(x) = \begin{cases} \sqrt{2}\, r_{2n} & \text{if } x = r_n \\ \sqrt{2}\, r_{2n+1} & \text{if } x = \sqrt{2}\, r_{2n} \\ x & \text{otherwise} \end{cases}$$

Is f one-to-one? No.

$$f(\sqrt{2}\, r_{2n}) = \sqrt{2}\, r_{2n+1} = f(\sqrt{2}\, r_{2n+1})$$

Draw picture to help



every number has an arrow showing where f maps it but no number has more than one arrow leading to it.

Thus the function is one-to-one.

But, how to describe it nicer and in a way that is understandable.

Can it be made easier by including

$$\mathcal{B} = \{\sqrt{3}r_1, \sqrt{3}r_2, \sqrt{3}r_3, \sqrt{3}r_4, \dots\} = \{\sqrt{3}r_n : n \in \mathbb{N}\}$$

What about defining

$$A_p = \{\sqrt{p}\, r_n : n \in \mathbb{N}\}$$

for every prime number p?

Then $A_p \cap A_q = \emptyset$ if $p \neq q$.

Alternatively,

<u>Hint 3</u> $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$.

These are enough hints to solve the problem in a nice way. Another approach...

Look for a book that has this problem solved in it. For example, look in the online texts linked from the webpage or the supplemental text in the bookstore.

What is the difference between

$$\limsup_{n\to\infty} x_n \quad \text{and} \quad \lim_{n\to\infty} x_n ?$$

Nothing if $\lim_{n\to\infty} x_n$ exists, but if $\lim_{n\to\infty} x_n$ doesn't exist then $\limsup_{n\to\infty} x_n$ still does.

We proved this last time. It is also clear from the definition

$$\limsup_{n\to\infty} x_n = \sup\{x \in \mathbb{R}^* : x \text{ is a cluster point of } x_n\}.$$

The completeness axiom states that the supremum of any set $A \subseteq \mathbb{R}^*$ is some number in $\mathbb{R}^*$.

Did anyone plot the sequence

$$x_n = \sin n$$

on the number line?

could someone do this for next time?

Consider the sequence

$$x_n = n.$$

what are the cluster points of this sequence?

x is a cluster point if for some subsequence x_{n_k} we have that $x_{n_k} \to x$.

Clearly for every subsequence

$$x_{n_k} \to \infty$$

Thus the set of cluster points is $\{\infty\}$.

Hence

$$\limsup_{n\to\infty} x_n = \sup\{\infty\} = \infty.$$

Note that it is important to include ∞ in the set of cluster points when computing suprema because $\sup \emptyset = -\infty$.

Consider the sequence

$$x_n = n \sin n .$$

Is $x = \infty$ a cluster point?

Define

$$J = \{k : \sin k > \tfrac{1}{2}\} = \{n_1, n_2, n_3, \dots\}$$

where $n_1 < n_2 < n_3 < \cdots$.

Then the subsequence

$$x_{n_k} = n_k \sin n_k > \tfrac{1}{2} n_k \to \infty$$

and so ∞ is a cluster point.

Is there anything that needs to be checked in the above argument?

What if we define

$$J = \{k : \sin k > \tfrac{9}{10}\}$$

does it still work?

What about

$$J = \{k : \sin k > 103\} \; ?$$

Clearly writing

$$\mathcal{J} = \{n_1, n_2, n_3, \ldots\} \qquad n_1 < n_2 < n_3 < \ldots$$

to define a subsequence requires that $\mathcal{J}$ be infinite. In particular $\mathcal{J} = \emptyset$ is bad.

Claim $\mathcal{J} = \{k : \sin k > \frac{1}{2}\}$ is infinite.

How to prove this?

Ideas?

The techniques discoverd while proving $\mathcal{J}$ is infinite might help with the extra credit problem to check whether the set of cluster points for the sequence $A_n = \sin n$ is equal or not to the interval $[-1,1]$.

Additional extra credit:

Let $B = \{n \sin n : n \in \mathbb{N}\}$. True or False that the closure

$$\overline{B} = \mathbb{R} \,?$$